ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Each proposed problem or solution should be submitted on a separate sheet or sheets, preferably typed in double spacing, in the format used below, to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106.

Solutions should be received within four months of the publication date of the proposed problem. Contributors in the United States who desire acknowledgement of receipt of contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS

$$F_0 = 0,\ \ F_1 = 1,\ \ F_{n+2} = F_{n+1} + F_n;\ \ L_0 = 2,\ \ L_1 = 1,\ \ L_{n+2} = L_{n+1} + L_n.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-256 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $L_{2n} - 3(-1)^n$ is the product of two Lucas numbers.

B-257 Proposed by Herta T. Freitag, Roanoke, Virginia.

Show that $[L_{2n} + 3(-1)^n]/5$ is the product of two Fibonacci numbers.

B-258 Proposed by Paul Bruckman, University of Illinois, Chicago, Illinois.

Let $[x]$ denote the greatest integer in x, $a = (1 + \sqrt{5})/2$, and $e_n = \{1 + (-1)^n\}/2$. Prove that for all positive integers m and n:

a. $$nF_{n+1} = [naF_n] + e_n,$$

b. $$nF_{m+n} = F_m\{[naF_n] + e_n\} + nF_{m-1}F_n .$$

B-259 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Characterize the infinite sequence of ordered pairs of integers (m, r), with $4 \le 2r \le m$, for which the three binomial coefficients

$$\binom{m-2}{r-2},\quad \binom{m-2}{r-1},\quad \binom{m-2}{r}$$

are in arithmetic progression.

B-260 Proposed by John L. Hunsucker and Jack Nebb, University of Georgia, Athens, Georgia.

Let $\sigma(n)$ denote the sum of the positive integral divisors of n. Show that $\sigma(mn) > \sigma(m) + \sigma(n)$ for all integers $m > 1$ and $n > 1$.

B-261 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let d be a positive integer and let S be the set of all nonnegative integers n such that $2^n - 1$ is an integral multiple of d. Show that either $S = \{0\}$ or the integers in S form an infinite arithmetic progression.

SOLUTIONS

SLIGHT MISPRINT, OTHERWISE FINE

B-232 Proposed by Guy A. R. Guillotte, Quebec, Canada.

In the following multiplication alphametic, the five letters F, Q, I, N, and E represent distinct digits. The dashes denote not necessarily distinct digits. What are the digits of FINE FQ?

```
  FQ
  FQ
 ---
 ---
----
FINE
```

(The number of dashes has been corrected.)

Solution by Ralph Garfield, The College of Insurance, New York, New York.

We first observe that F must be 9. Furthermore, Q must be 5 or more since $94^2 = 8836$, which does not give a first digit of 9. Clearly, since E and Q are distinct, Q cannot be 5 or 6. Also, since F and Q are different, Q cannot be 9. Hence Q must be 8. Then $98^2 = 9604$ gives 9604 98 as FINE FQ.

Also solved by Sister Marion Beiter, Ashok K. Chandra, J. A. H. Hunter, John W. Milsom, Charles W. Trigg, David Zeitlin, and the Proposer.

A FIBONACCI QUADRATIC

B-233 Proposed by Harlan L. Umansky, Emerson High School, Union City, New Jersey.

Show that the roots of $F_{n-1}x^2 - F_nx - F_{n+1} = 0$ are $x = -1$ and $x = F_{n+1}/F_{n-1}$. Generalize to show a similar result for all sequences formed in the same manner as the Fibonacci sequence.

Solution by Graham Lord, S. U. N. Y., Binghamton, New York.

Let a_n be any sequence satisfying $a_n = a_{n-1} + a_{n-2}$, for all n. Then

$$P(x) = a_{n-1}x^2 - a_n x - a_{n+1} = a_{n-1}x - (a_{n+1} - a_{n-1})x - a_{n+1}$$
$$= (a_{n-1}x - a_{n+1})(x + 1) .$$

Hence the roots of $P(x) = 0$ are -1 and a_{n+1}/a_{n-1}. In particular, if $a_n = F_n$ the first half of the question is also solved.

Also solved by Sister Marion Beiter, Paul S. Bruckman, Ashok K. Chandra, Herta T. Freitag, Ralph Garfield, J. A. H. Hunter, Edgar Karst, Peter A. Lindstrom, Graham Lord, John W. Milsom, Paul Salomaa, David Zeitlin, and the Proposer.

DUPLICATING A CUBE?

B-234 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California

Prove that

$$L_n^3 = 2F_{n-1}^3 + F_n^3 + 6F_{n-1}F_{n+1}^2 .$$

Solution by Paul S. Bruckman, University of Illinois, Chicago, Illinois.

Since $F_n = F_{n+1} - F_{n-1}$, we may cube both sides and obtain

$$F_n^3 = F_{n+1}^3 - 3F_{n+1}^2 F_{n-1} + 3F_{n+1}F_{n-1}^2 - F_{n-1}^3 .$$

Adding $2F_{n-1}^3 + 6F_{n-1}F_{n+1}^2$ to both sides of this expression, we obtain

$$2F_{n-1}^3 + F_n^3 + 6F_{n-1}F_{n+1}^2 = F_{n+1}^3 + 3F_{n+1}^2 F_{n-1} + 3F_{n+1}F_{n-1}^2 + F_{n-1}^3 = (F_{n+1} + F_{n-1})^3 = L_n^3.$$

Also solved by James D. Bryant, Ashok K. Chandra, Warren Cheves, Herta T. Freitag, Ralph Garfield, J. A. H. Hunter, Edgar Karst, Peter A. Lindstrom, Graham Lord, John W. Milsom, Paul Salomaa, David Zeitlin, and the Proposer.

A PROPERTY OF F_{16}

B-235 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the largest positive integer n such that F_n is smaller than the sum of the cubes of the digits of F_n.

Solution by Ashok K. Chandra, Graduate Student, Stanford University, California.

We need only check for all n such that $F_n \le 4 \cdot 9^3 = 2916$, for if F_n has n digits, $n \ge 5$, then $n \cdot 9^3 < 10^{n-1} \le F_n$.

$n =$	15	16	17	18
$F_n =$	610	987	1597	2584

Now,

$$2584 > 2^3 + 5^3 + 8^3 + 4^3 ,$$

and

$$1597 > 1^3 + 5^3 + 9^3 + 7^3 ,$$

but

$$987 < 9^3 + 8^3 + 7^3 .$$

Hence the largest n is 16, and $F_n = 987$.

I wrote a short computer program to determine the largest n for an arbitrary power, and obtained the following results:

Power	n	F_n
2	11	89
3	16	987
4	19	4,181
5	24	46,368
6	29	514,229
7	34	5,702,887
8	39	63,245,986
9	42	267,914,296

Also solved by Paul S. Bruckman, Paul Salomaa, Charles W. Trigg, and the Proposer.

TWO HEADS NOT BETTER THAN ONE

B-236 Proposed by Paul S. Bruckman, San Rafael, California.

Let P_n denote the probability that, in n throws of a coin, two consecutive heads will not appear. Prove that

$$P_n = 2^{-n}F_{n+2} .$$

Solution by J. L. Brown, Jr., Pennsylvania State University, Pennsylvania.

A sequence of H's and T's of length n is called <u>admissible</u> if two heads do not appear together anywhere in the sequence. Let α_n be the number of admissible sequences of length n. Then there are α_n admissible sequences of length $n + 1$ which end with a T, while there are α_{n-1} admissible sequences of length $n + 1$ ending in an H (since any such sequence must actually end in TH). Thus $\alpha_{n+1} = \alpha_n + \alpha_{n-1}$. Combined with the initial values $\alpha_1 = 2$ and $\alpha_2 = 3$, we find $\alpha_n = F_{n+2}$ for $n \geq 1$ and the required probability becomes $F_{n+2}/2^n$ as stated.

NOTE: Essentially the same problem occurs as Problem 62-6 in SIAM Review (solution in Vol. 6, No. 3, July 1964, p. 313) and as Problem E2022 in American Mathematical Monthly (solution in Vol. 74, No. 10, December, 1968, p. 1117). See also Problem 1, p. 14 in An Introduction to Combinatorial Analysis by John Riordan (J. Wiley and Sons, Inc., 1958) and Problem B-5 in the Fibonacci Quarterly (solution in Vol. 1, No. 3, October, 1963, p. 79).

Also solved by Ashok K. Chandra, Ralph Garfield, Peter A. Lindstrom, Graham Lord, Bob Prielipp, Paul Salomaa, Richard W. Sielaff, and the Proposer.

G.C.D. PROBLEM

B-237 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let (m,n) denote the greatest common divisor of the integers m and n.

(i) Given $(a,b) = 1$, prove that $(a^2 + b^2, a^2 + 2ab)$ is 1 or 5.

(ii) Prove the converse of Part (i).

Solution by Paul Salomaa, Junior, M. I. T., Cambridge, Massachusetts.

$$(a^2 + b^2,\ a^2 + 2ab) = (a^2 + b^2 - [a^2 + 2ab],\ a^2 + 2ab) = (b^2 - 2ab,\ a^2 + 2ab)\ .$$

Let p be a prime (or 1) such that

$$p \mid (a^2 + b^2,\ a^2 + 2ab)\ .$$

Then

$$p \mid (b^2 - 2ab) \quad \text{and} \quad p \mid (a^2 + 2ab)\ .$$

But $(p,a) = (p,b) = 1$, for if not, $p \mid (a^2 + b^2)$ would imply $p \mid (a,b)$ forcing $p = 1$. Hence $p \mid (b - 2a)$ and $p \mid (a + 2b)$. So

$$p \mid [\,b - 2a + 2(a + 2b)\,]\ ,$$

i.e., $p \mid 5b$. Since $(p,b) = 1$, we have that $p \mid 5$, so $p = 5$ or $p = 1$. If

$$p^2 \mid (a^2 + b^2,\ a^2 + 2ab)\ ,$$

then

$$p^2 \mid (b - 2a) \quad \text{and} \quad p^2 \mid (a + 2b)\ ,$$

hence $p^2 \mid 5b$, so in this case $p = 1$. In particular, $5^2 \nmid (a^2 + b^2, a^2 + 2ab)$. For the converse, let $(a,b) = d$. Then $d^2 \mid (a^2 + b^2, a^2 + 2ab)$. Hence $d^2 \mid 5$ or $d^2 \mid 1$, and in either case, $d = 1$.

Also solved by Ashok K. Chandra, Herta T. Freitag, Graham Lord, and the Proposer.