A GEOMETRIC TREATMENT OF SOME OF THE ALGEBRAIC PROPERTIES OF THE GOLDEN SECTION

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Our object is to present a single geometric setting in which it is possible to deduce some of the more familiar algebraic properties of the golden section, ϕ. In this setting, we also uncover some less familiar properties of ϕ and some extensions and generalizations of the "golden" sequence:

$$1,\ \phi,\ \phi^2,\ \phi^3,\ \cdots,\ \phi^n,\ \cdots\ .$$

The setting in which we will work is motivated by consideration of the following problem: construct a semi-circle on a given setment $\overline{AB}$. Locate a point P on the semi-circle so that the length of the projection of $\overline{PA}$ on $\overline{AB}$ is equal to PB. (See Fig. 1.) Since in right triangle APB, PB is the mean proportional between AB and GB, and since AG = PB, we conclude that G is the golden section of $\overline{AB}$. (For a more familiar construction of G, cf. [1].)

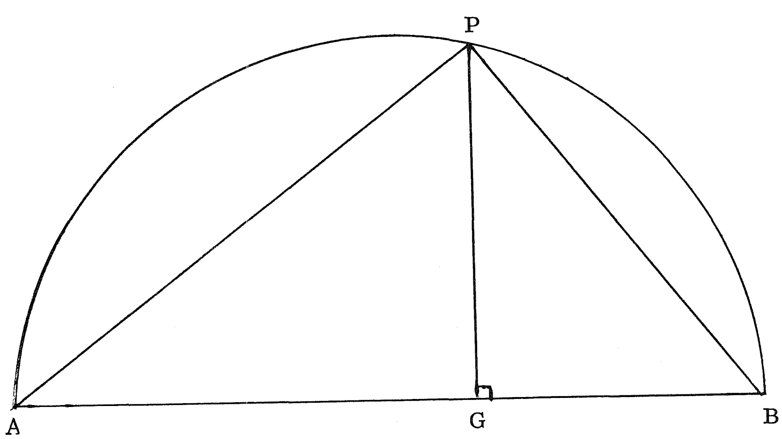

Figure 1

The right triangle APB is not "golden." In fact, if we normalize by taking GB = 1, then AG = PB = $(1 + \sqrt{5})/2$, which as usual, we denote by ϕ. Since $\overline{PG}$ is easily seen to have length $\sqrt{\phi}$, we deduce from right triangle PGB the property that $\phi^2 = \phi + 1$. Using this property in conjunction with right triangle APB, we conclude that PA = $\phi^{3/2}$. So, we see that the ratio of the legs of right triangle APB is $\sqrt{\phi}$. Nevertheless, this normalized right triangle is the one with which we shall work.

At B construct a perpendicular on the same side of $\overline{AB}$ as P and extend $\overline{AP}$ until it meets the perpendicular at P_2. At P_2 construct a perpendicular meeting the extension of $\overline{AB}$ at B_2. Set $P_1 = P$, $B_0 = G$, and $B_1 = B$. (See Fig. 2.)

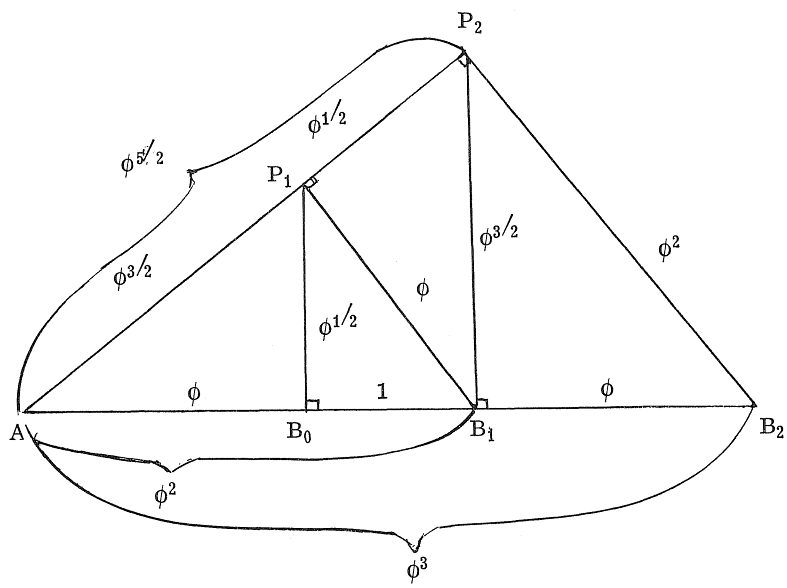

Figure 2

From above, we have that $AB_0 = \phi$, $B_0B_1 = 1$, $AB_1 = \phi^2$, $P_1B_1 = \phi$, $AP_1 = \phi^{3/2}$. The following lengths are easily deduced from these, the Pythagorean theorem, and similarity arguments: $P_1P_2 = \phi^{1/2}$, $P_2B_1 = \phi^{3/2}$, $AP_2 = \phi^{5/2}$, $B_1B_2 = \phi$, $P_2B_2 = \phi^2$. $AB_2 = AB_1 + B_1B_2 = \phi^2 + \phi = \phi(\phi + 1) = \phi^3$. (See Fig. 2.)

Since $AP_2 = AP_1 + P_1P_2$, and $AB_2 = AB_0 + B_0B_1 + B_1B_2$, we deduce (purely geometrically) two more properties: $\phi^{5/2} = \phi^{3/2} + \phi^{1/2}$, $\phi^3 = 2\phi + 1$.

The procedure for continuing now is clear: in the same manner as we constructed triangle AP_2B_2 from triangle AB_1P_1, we construct a new triangle AP_3B_2 from AP_2B_2, and so on. That is, we can generate a sequence of right triangles AP_1B_1, AP_2B_2, $\cdots$, AP_nB_n, $\cdots$ having the following characteristics:

(i) $P_nB_n = \phi^n, \quad n = 1, 2, 3, \cdots$

(ii) $AB_n = \phi^{n+1}, \quad n = 0, 1, 2, \cdots$

(iii) $AP_n = \phi^{(2n+1)/2}, \quad n = 1, 2, 3, \cdots$

(iv) $P_{n+1}B_n = \phi^{(2n+1)/2}, \quad n = 0, 1, 2, \cdots$

(v) $B_nB_{n+1} = \phi^n, \quad n = 0, 1, 2, \cdots$

(vi) $$P_nP_{n+1} = \phi^{(2n-1)/2}, \quad n = 1, 2, 3, \cdots$$

(vii) $$AB_n = AB_{n-1} + B_{n-1}B_n, \quad n = 1, 2, 3, \cdots,$$

implying $$\phi^{n+1} = \phi^n + \phi^{n-1}$$

(viii) $$AP_{n+1} = AP_n + P_nP_{n+1}, \quad n = 1, 2, 3, \cdots,$$

implying $$\phi^{(2n+3)/2} = \phi^{(2n+1)/2} + \phi^{(2n-1)/2} .$$

(ix)
$$\begin{array}{llll} AB_1 = AB_0 + B_0B_1, & \text{implying } \phi^2 = \phi + 1 \\ AB_2 = AB_1 + B_1B_2, & \text{implying } \phi^3 = 2\phi + 1 \\ AB_3 = AB_2 + B_2B_3, & \text{implying } \phi^4 = 3\phi + 2 \\ AB_4 = AB_3 + B_3B_4, & \text{implying } \phi^5 = 5\phi + 3 \\ \vdots \quad \vdots \quad \vdots & \quad \vdots \quad \vdots \end{array}$$

We note that the geometric result in (ix) demonstrates the equivalence of the geometric sequence $1, \phi, \phi^2, \phi^3, \cdots$ and the Fibonacci sequence $1, \phi, \phi + 1, 2\phi + 1, \cdots,$ implying that the sequence $1, \phi, \phi^2, \phi^3, \cdots$ is a "golden" sequence.

Having constructed a sequence of triangles "on the right" of the golden cut G in Fig. 1, we now construct a sequence "on the left." Drop a perpendicular from G to $\overline{AP}$, intersecting $\overline{AP}$ at P_0. From P_0 drop a perpendicular to $\overline{AB}$ intersecting $\overline{AB}$ at B_1'. Repeat this procedure once more, obtaining P_1' on $\overline{AP}$ and B_2' on $\overline{AB}$. Set $B_0 = G$, $P_1 = P$, and $B_1 = B$. (See Figure 3.) From above, we know that $AB_0 = \phi = P_1B_1$, $B_0B_1 = 1$, $AP_1 = \phi^{3/2}$, and $P_1B_0 = \phi^{1/2}$. Our object is to compute the lengths of the remaining segments in Fig. 3. The same kinds of arguments as above result in the following:
$P_0P_1 = \phi^{-1/2}$, $AP_0 = \phi^{1/2}$, $P_0B_0 = 1$, $B_1'B_0 = \phi^{-1}$, $AB_1' = 1$, $P_0B_1' = \phi^{-1/2}$, $P_1'P_0 = \phi^{-3/2}$, $AP_1' = \phi^{-1/2}$, $P_1'B_1' = \phi^{-1}$, $B_2'B_1' = \phi^{-2}$, $AB_2' = \phi^{-1}$, and $P_1'B_2' = \phi^{-3/2}$. (See Figure 3.)

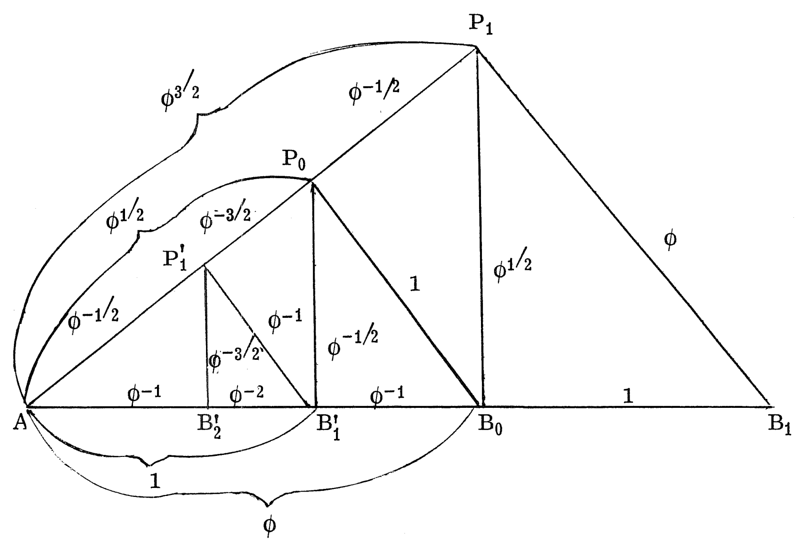

Figure 3

Again, the continuation procedure is clear: we can generate a sequence of right triangles AP_0B_0, $AP_1'B_1'$, $AP_2'B_2'$, $\cdots$, $AP_n'B_n'$, $\cdots$, with the following characteristics:

(i) $P_n'B_n' = \phi^{-n}$, $n = 0, 1, 2, \cdots$ (where $P_0'B_0' = P_0B_0$).

(ii) $AB_n' = \phi^{-n+1}$, $n = 0, 1, 2, \cdots$ (where $AB_0' = AB_0$).

(iii) $AP_n' = \phi^{-(2n-1)/2}$, $n = 0, 1, 2, \cdots$ (where $AP_0' = AP_0$).

(iv) $P_n'B_{n+1}' = \phi^{-(2n+1)/2}$, $n = 0, 1, 2, \cdots$ (where $P_0'B_1' = P_0B_1'$).

(v) $B_{n+1}'B_n' = \phi^{-(n+1)}$, $n = 0, 1, 2, \cdots$ (where $B_1'B_0' = B_1'B_0$).

(vi) $P_{n+1}'P_n' = \phi^{-(2n+3)/2}$, $n = 0, 1, 2, \cdots$ (where $P_1'P_0' = P_1'P_0$).

(vii) $AB_n' = AB_{n+1}' + B_{n+1}'B_n'$, $n = 0, 1, 2, \cdots$, implying

$$\phi^{-n+1} = \phi^{-n} + \phi^{-n-1}.$$

(viii) $AP_n' = AP_{n+1}' + P_{n+1}'P_n'$, $n = 0, 1, 2, \cdots$, implying

$$\phi^{-(2n-1)/2} = \phi^{-(2n+1)/2} + \phi^{-(2n+3)/2}.$$

(ix) $AB_0 = B_1'B_0 + B_2'B_1' + B_3'B_2' + \cdots + B_{n+1}'B_n' + \cdots$, implying

$$\phi = 1/\phi + 1/\phi^2 + 1/\phi^3 + \cdots + 1/\phi^{+(n+1)} + \cdots.$$

(x) $AP_0 = P_1'P_0 + P_2'P_1' + P_3'P_2' + \cdots + P_{n+1}'P_n' + \cdots$, implying

$$\phi^{1/2} = \phi^{-3/2} + \phi^{-5/2} + \phi^{-7/2} + \cdots + \phi^{-(2n+3)/2} + \cdots$$

Remark 1: A more familiar geometric setting for property (ix) is a rectangular spiral (cf. [1]) the length of whose n^{th} side is ϕ^{-n}. (See Fig. 4.) Figure 3 suggests that if one were to "unfold" the rectangular spiral onto a straight line, the union of the sides, $\overline{B_{n+1}'B_n'}$, would be a segment with length equal to AB_0. (An analogous remark can be made for property (x).)

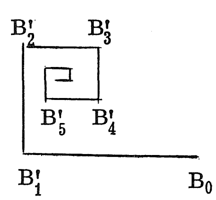

Figure 4

Remark 2: Figure 3 suggests the following generalization of properties (ix) and (x):

$$\phi^k = \phi^{k-2} + \phi^{k-3} + \cdots + \phi^{-1} + 1 + \phi + \phi^2 + \cdots,$$

for $k = 0, \pm 1/2, \pm 2/2, \pm 3/2, \cdots$.

In Figure 2, we see that the "golden sequence," $1, \phi, \phi^2, \cdots, \phi^n, \cdots$ has its geometric analogue in the sequence of altitudes $P_0B_0,\ P_1B_1,\ P_2B_2, \cdots, P_nB_n, \cdots$. In that same figure, the sequence of altitudes $P_1B_0,\ P_2B_1,\ P_3B_2, \cdots, P_{n+1}B_n, \cdots$ suggests a second sequence which is also golden: $\phi^{1/2},\ \phi^{3/2},\ \phi^{5/2}, \cdots$. (That this sequence is geometric is clear; property (viii) demonstrates that it is also a Fibonacci sequence.) The following are additional extensions of the golden sequence suggested by the appropriate sequences of altitudes in Figs. 2 and 3 (we include the above two for completeness):

(1) $1, \phi, \phi^2, \phi^3, \cdots, \phi^{n-1}, \cdots$ (Golden Sequence)

(2) $1/\phi, 1/\phi^2, 1/\phi^3, \cdots, 1/\phi^n, \cdots$

(3) $\cdots, 1/\phi^n, 1/\phi^{n-1}, \cdots, 1/\phi, 1, \phi, \phi^2, \phi^3, \cdots, \phi^{n-1}, \cdots$

(4) $\phi^{1/2}, \phi^{3/2}, \phi^{5/2}, \cdots, \phi^{(2n-1)/2}, \cdots$

(5) $\phi^{-1/2}, \phi^{-3/2}, \phi^{-5/2}, \cdots, \phi^{-(2n-1)/2}, \cdots$

(6) $\cdots \phi^{-(2n-1)/2}, \cdots, \phi^{-1/2}, \phi^{1/2}, \phi^{3/2}, \phi^{5/2}, \cdots, \phi^{(2n-1)/2}, \cdots$

As a final remark, we consider the sequence suggested by the complete sequence of altitudes in Fig. 3:

$$\cdots, P'_nB'_{n+1},\ P'_nB'_n,\ P'_{n-1}B'_n,\ P'_{n-1}B'_{n-1}, \cdots, P'_1B'_2,\ P'_1B'_1,\ P_0B'_1,\ P_0B_0,$$
$$P_1B_0,\ P_1B_1,\ \cdots,\ P_nB_{n-1},\ P_nB_n, \cdots .$$

This geometric sequence, with ratio $\phi^{1/2}$, is evidently

$$\cdots, \phi^{-(2n+1)/2},\ \phi^{-n},\ \phi^{-(2n-1)/2},\ \phi^{-(n-1)}, \cdots, \phi^{-3/2},\ \phi^{-1},\ \phi^{-1/2},\ 1,$$
$$\phi^{1/2},\ \phi, \cdots, \phi^{(2n-1)/2},\ \phi^n, \cdots .$$

Although this is not a Fibonacci sequence (and Hence, not golden), it contains each of the golden sequences, (1)-(6), as subsequences, and has the easily verified property that any subsequence consisting of alternate terms of the sequence, is in fact, a golden sequence.

REFERENCE

1. H. E. Huntley, The Divine Proportion, Dover, New York, 1970.

ERRATA

In "Ye Olde Fibonacci Curiosity Shoppe," appearing in Vol. 10, No. 4, October, 1972, please make the following changes:

Page 443: In the first line of the second paragraph, insert the word "ten" between "of" and "consecutive," so that it reads "... the sum of the squares of ten consecutive Fibonacci numbers is always divisible by $F_{10} = 55$."