n-FIBONACCI PRODUCTS

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1. NOTATION

Let ϕ^n be the n-dimensional vector space, i.e., for

$$X = [x_1, x_2, \cdots, x_n] \in \phi^n, \qquad x_1, x_2, \cdots, x_n \in \phi^1 .$$

In addition, let I be the set of positive integers, J the set of non-negative integers, $I(n) \subset I$, be such that if $k \in I(n)$ then $k \le n$, $J(n) \subset J$, be such that if $k \in J(n)$ then $k \le n$. $W(n) \subset \phi^n$ is such that if $K \in W(n)$, where $K = [k_1, k_2, \cdots, k_n]$, then $k_m \in J$, for $m \in I(n)$. In particular, $U = [1, 1, \cdots, 1] \in W(n)$.

With $K \in W(n)$ and $X \in \phi^n$, we write

$$X^K = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} = \prod_{m=1}^{n} x_m^{k_m} , \tag{1}$$

and in particular

$$X^U = x_1 x_2 \cdots x_n . \tag{2}$$

Also

$$|X| = \sum_{m=1}^{n} x_m , \tag{3}$$

and

$$\sum_{K=0}^{P} f(K) \tag{4}$$

is the sum of all elements of the form $f(K)$ where the component of K, i.e., k_m, $m \in I(n)$, take all integer values such that $0 \le k_m \le p_m$, where $P = [p_1, p_2, \cdots, p_n] \in W(n)$.

Let $E(m)$ be the partial translation operator for the variable x_m, i.e.,

$$E(m) f(x_k) = \delta_m^k f(x_k + 1) , \qquad k, m \in I(n) , \tag{5}$$

where δ_k^m is the Kronecker delta. In addition, let $\Delta(m) = E(m) - Id$, Id being the identity operator. Using the vector notation introduced earlier, we have

(6) $$E = [E(1), E(2), \cdots, E(n)]$$

(7) $$\Delta = [\Delta(1), \Delta(2), \cdots, \Delta(n)]$$

and

$$E^U = E(1)E(2)\cdots E(n), \qquad \Delta^U = \Delta(1)\Delta(2)\cdots\Delta(n) .$$

2. FIBONACCI AND LUCAS PRODUCTS

Let $F(m)$ be the general term of the Fibonacci sequence, $L(m)$ the general term of the Lucas sequence as defined in [1] and $H(m)$ the general term of the generalized Fibonacci sequence. Using the notation introduced in Section 1, we have with

$$K = [k_1, k_2, \cdots, k_n] \in W(n)$$

(8) $$F(K) = [F(k_1), F(k_2), \cdots, F(k_n)]$$

(9) $$L(K) = [L(k_1), L(k_2), \cdots, L(k_n)]$$

(10) $$H(K) = [H(k_1), H(k_2), \cdots, H(k_n)]$$

and

(11) $$f(K) = [F(K)]^U = \prod_{m=1}^{n} F(k_m)$$

(12) $$\lambda(K) = [L(K)]^U = \prod_{m=1}^{n} L(k_m)$$

(13) $$h(K) = [H(K)]^U = \prod_{m=1}^{n} H(k_m) .$$

The numbers $f(K)$, $\lambda(K)$, and $h(K)$ are called the n-Fibonacci, Lucas and generalized Fibonacci products.

3. RECURRENCE RELATIONS

According to [1] we have for the three sequences considered

(14) $$F(k_m + 2) = F(k_m + 1) + F(k_m)$$

which we can write

(15) $$E(m)\Delta(m)F(k_m) = F(k_m)$$

or

(16) $$[E(m)\Delta(m) - Id]F(k_m) = 0 .$$

Starting from (15) we can write

$$\prod_{m=1}^{n} F(m)\Delta(m)F(k_m) = \prod_{m=1}^{n} F(k_m),$$

or

$$E^U \Delta^U f(K) = f(K),$$

or again

(17) $$(E^U \Delta^U - Id)f(K) = 0.$$

Thus the Fibonacci products satisfy a recurrence relation similar to the one dimension, i.e., (16). The same applies to the Lucas and generalized Fibonacci products, i.e.,

(18) $$(E^U \Delta^U - Id)\lambda(K) = 0,$$

(19) $$(E^U \Delta^U - Id)h(K) = 0.$$

4. OTHER RELATIONS

The relations given in [1, pp. 59-60] can be generalized for n-Fibonacci and Lucas products. We illustrate by two examples:

(i) Relation (I 14) reads: $L(m) = F(m + 2) - F(m - 2)$, or

$$L(m + 2) = F(m + 4) - F(m)$$

or, on operator form

(20) $$E^2(m)L(m) = [E^4(m) - Id]F(m).$$

But

$$E^4(m) - Id = [E(m) - Id][E(m) + Id][E^2(m) + Id],$$

where $E(m) - Id = \Delta(m)$.

$$E(m) + Id = 2M(m)$$

where $M(m)$ is the partial mean operator. We define correspondingly

$$M = [M(1), M(2), \cdots, M(n)],$$

and

$$M^U = \prod_{m=1}^{n} M(m).$$

In addition let

$$P(m) = E^2(m) + Id, \qquad P = [P(1), P(2), \cdots, P(n)],$$

and

$$P^U = P(1)P(2) \cdots P(n) = \prod_{m=1}^{n} P(m) .$$

We take now the product of both sides of (20) which we rewrite

$$\prod_{m=1}^{n} E^2(m)L(k_m) = \prod_{m=1}^{n} 2\Delta(m)M(m)P(m)F(k_m) \tag{21}$$

or

$$E^{2U}\lambda(K) = 2^n\Delta^U M^U P^U f(K) , \tag{22}$$

which is the relation corresponding to (I 14) of [1] for n-Fibonacci and Lucas products.

(ii) Relation (I 41) can be written

$$\sum_{k=0}^{2q} \binom{2q}{k} F(2k + p) = 5^q F(2q + p) ,$$

or, introducing the variable m and the usual operators

$$\sum_{k_m=0}^{2q_m} \binom{2q_m}{k_m} E^{2k_m}(m)F(p_m) = 5^{q_k} E^{2q_m}(m)F(p_m) . \tag{23}$$

Taking the product over m from $m = 1$ to $m = n$ and using the notation

$$\prod_{m=1}^{n} \binom{2q_m}{k_m} = \binom{2Q}{K},$$

where

$$K = [k_1, k_2, \cdots, k_n] \in W(n) , \qquad Q, P \in W(n) ,$$

we obtain the formula corresponding to (I 41), i.e.,

$$\left[\sum_{K=0}^{2Q} \binom{2Q}{K} E^{2K} - 5^{|Q|} E^{2Q}\right] f(P) = 0 . \tag{24}$$

REFERENCE

1. V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, New York, 1969.