PERIODICITY OF SECOND- AND THIRD-ORDER RECURRING SEQUENCES

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Define a sequence of generalized Fibonacci numbers

$$\{w_n\}_0^\infty = \{w_n(b,c;\ P,Q)\}_0^\infty \tag{1}$$

by

$$w_n = bw_{n-1} + cw_{n-2}, \tag{2}$$

where n denotes an integer ≥ 2, $w_0 = P$ and $w_1 = Q$. Considering a special form of this sequence

$$\{w_n^{(1)}\}_0^\infty = \{w_n(1,1;\ 0,1)\}_0^\infty ,$$

D. D. Wall [1] has shown that

$$\{w_n^{(1)} \pmod{m}\}_0^\infty$$

(where m denotes a positive integer) is simply periodic. Our objective is to point out a rigorous proof of the same and extend it to the sequence of Tribonacci numbers

$$\{T_n\}_0^\infty = \{T_n(b,c,d;\ P,Q,R)\}_0^\infty . \tag{3}$$

This sequence of numbers is defined by

$$T_n = bT_{n-1} + cT_{n-2} + dT_{n-3}, \tag{4}$$

where n denotes an integer ≥ 3, $T_0 = P$, $T_1 = Q$ and $T_2 = R$.

Theorem a.

$$\{w_n^{(1)} \pmod{m}\}_0^\infty$$

is simply periodic.

Proof. Let

$$m = \Pi p_j^{a_j} ,$$

where $j = 1, 2, \cdots, i$ and p_j represents a prime. Since

$$\{ w_n^{(1)} \pmod{p_j^{a_j}} \}_0^\infty$$

is known to be periodic [1], we denote the length of the period

$$\{ w_n^{(1)} \pmod{p_j^{a_j}} \}_0^\infty$$

by k_j and write

$$w_{k_j}^{(1)} \equiv 0 \pmod{p_j^{a_j}} , \qquad w_{k_j+1}^{(1)} \equiv 1 \pmod{p_j^{a_j}} . \tag{5}$$

Then it is easy to show that

$$w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{p_1^{a_1}}, \quad w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{p_2^{a_2}} , \cdots , \quad w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{p_i^{a_i}}$$

(6) and

$$w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{p_1^{a_1}}, \quad w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{p_2^{a_2}} , \cdots , \quad w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{p_i^{a_i}} .$$

Therefore, it follows that

$$w_{k_1k_2\cdots k_i}^{(1)} \equiv 0 \pmod{m}$$

(7) and

$$w_{k_1k_2\cdots k_i+1}^{(1)} \equiv 1 \pmod{m}$$

and

$$\{ w_n^{(1)} \pmod{m} \}_0^\infty$$

becomes simply periodic.

Theorem b. If $(b, c, P, Q, m) = 1$, then $\{w_n \pmod m\}_0^\infty$ is simply periodic.

Proof. Let

$$\{w_n^{(2)}\}_0^\infty = \{w_n(b, c; 0, 1)\}_0^\infty .$$

For p denoting a prime, if $(b,c,p) = 1$, then it has been shown in [3], that $\{w_n^{(2)} \pmod p\}_0^\infty$ is simply periodic. Also, since

$$w_n = pw_n^{(2)} + cQw_{n-1}^{(2)} ,$$

it follows that if $(b,c,P,Q,p) = 1$, then $\{w_n \pmod p\}_0^\infty$ is simply periodic, and the technique of Theorem a renders that $\{w_n \pmod m\}_0^\infty$ is simply periodic.

Theorem c. Let

$$\{T_n^{(9)}\}_0^\infty = \{T_n(1, 1, 1; 0, 0, 1)\}_0^\infty .$$

Then

$$\{T_n^{(9)} \pmod m\}_0^\infty$$

is simply periodic.

Proof. We have shown in [2], that $\{T_n^{(9)} \pmod p\}_0^\infty$ is simply periodic and the proof that $\{T_n^{(9)} \pmod m\}_0^\infty$ is simply periodic follows from the technique of Theorem a.

Theorem d. If $(b, c, d, P, Q, R, m) = 1$, then $\{T_n \pmod m\}_0^\infty$ is simply periodic.

The proof of this theorem is similar to that of Theorem c and is left to the reader.

ACKNOWLEDGEMENT

I am grateful to Dr. Joseph Arkin for drawing my attention to the problem of this note in connection with an earlier paper.

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