EXPONENTIAL GENERATING FUNCTIONS FOR FIBONACCI IDENTITIES

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1. INTRODUCTION

Generating functions provide a starting point for an apprentice Fibonacci enthusiast who would like to do some research. In the Fibonacci Primer: Part VI, Hoggatt and Lind [1] discuss ordinary generating functions for identities relating Fibonacci and Lucas numbers. Also, Gould [2] has worked with generalized generating functions. Here, we use exponential generating functions to establish some Fibonacci and Lucas identities.

2. THE EXPONENTIAL FUNCTION AND EXPONENTIAL GENERATING FUNCTIONS

The exponential function e^x appears in studying radioactive decay, bacterial growth, compound interest, and probability theory. The transcendental constant $e \doteq 2.718$ is the base for natural logarithms. However, the particular property of e^x that interests us is

$$(1) \qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

Then

$$e^{\alpha t} = 1 + \frac{\alpha t}{1!} + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^4}{4!} + \cdots$$

and algebra shows that

$$(2) \qquad e^{\alpha t} - e^{\beta t} = (1 - 1) + \frac{(\alpha - \beta)t}{1!} + \frac{(\alpha^2 - \beta^2)t^2}{2!} + \frac{(\alpha^3 - \beta^3)t^3}{3!} + \cdots .$$

To relate (2) to Fibonacci numbers, if F_n is the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, and if $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, then it is well known that

$$(3) \qquad F_n = (\alpha^n - \beta^n)/(\alpha - \beta) .$$

Thus, dividing Eq. (2) by $(\alpha - \beta)$ gives

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \frac{F_1 t}{1!} + \frac{F_2 t^2}{2!} + \frac{F_3 t^3}{3!} + \frac{F_4 t^4}{4!} + \cdots = \sum_{n=1}^{\infty} F_n \frac{t^n}{n!} .$$

since $F_0 = 0$, we can add the term $F_0 \frac{t^0}{0!}$ and write the following exponential generating function for Fibonacci numbers:

$$\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \quad . \tag{4}$$

An elementary companion to the Fibonacci exponential generating function generates Lucas number coefficients. The Lucas numbers are defined by $L_1 = 1$, $L_2 = 3$, $L_n + L_{n-1} = L_{n+1}$, and have the property that

$$L_n = \alpha^n + \beta^n \quad . \tag{5}$$

If the power series for $e^{\alpha t}$ and $e^{\beta t}$ are calculated and then added term-by-term, the result is

$$e^{\alpha t} + e^{\beta t} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!} \quad . \tag{6}$$

For a novel use for these elementary generating functions, the reader is directed to [3] for a proof that the determinant of e^{Q^n} is e^{L_n}, where $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

3. PROPERTIES OF INFINITE SERIES

We list without proof some properties of infinite series necessary to our development of exponential generating functions.

Given

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \quad ,$$

it follows that

$$A(t)\,B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!} \quad ,$$

$$A(t)\,B(-t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k b_{n-k} \right) \frac{t^n}{n!} \quad . \tag{7}$$

Thus, if $B(t) = e^t$, then $b_n = 1$ for all n, and

$$A(t)\,e^t = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k \right) \frac{t^n}{n!} \quad .$$

To help the reader with the double summation notation, let

$$A(t) = \sum_{n=0}^{\infty} n \frac{t^n}{n!} \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,.$$

Then

$$\begin{aligned}
A(t)\,B(t) &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}k\right)\frac{t^n}{n!}\\
&= \left(\sum_{k=0}^{0}\binom{0}{k}k\right)\frac{t^0}{0!} + \left(\sum_{k=0}^{1}\binom{1}{k}k\right)\frac{t^1}{1!} + \left(\sum_{k=0}^{2}\binom{2}{k}k\right)\frac{t^2}{2!} + \cdots\\
&= \binom{0}{0}0\frac{t^0}{0!} + \left(\binom{0}{0}0 + \binom{1}{1}1\right)\frac{t^1}{1!} + \left(\binom{2}{0}0 + \binom{2}{1}1 + \binom{2}{2}2\right)\frac{t^2}{2!} + \cdots\\
&= 0 + \frac{t}{1!} + \frac{4t^2}{2!} + \cdots + te^{2t} = \sum_{n=0}^{\infty}\frac{t(2t)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n t^{n+1}}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(n+1)2^n t^{n+1}}{(n+1)!}\\
&= \sum_{n=0}^{\infty}\frac{(n2^{n-1})t^n}{n!}
\end{aligned}$$

where $\binom{n}{k}$ is the binomial coefficient,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,.$$

4. EXPONENTIAL GENERATING FUNCTIONS FOR FIBONACCI IDENTITIES

Generating function (4) and algebraic properties of α and β, the roots of $x^2 - x - 1 = 0$, give us an easy way to generate Fibonacci identities. Useful algebraic properties of $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ include:

$$\begin{array}{lll}
\alpha\beta = -1 & \alpha^2 = \alpha + 1 & F_n = (\alpha^n - \beta^n)/(\alpha - \beta)\\
\alpha - \beta = \sqrt{5} & \alpha^m = \alpha F_m + F_{m-1} & L_n = \alpha^n + \beta^n\\
\alpha + \beta = 1 & &
\end{array}\,.$$

Take $B(t) = e^t$ and $A(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta)$. (See Eqs. (1) and (4).) Then their series product $A(t)$ and $B(t)$ gives

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k\right)\frac{t^n}{n!} = \frac{e^{(\alpha+1)t} - e^{(\beta+1)t}}{\alpha - \beta} = \frac{e^{\alpha^2 t} - e^{\beta^2 t}}{\alpha - \beta}$$

(8)

$$= \sum_{n=0}^{\infty} F_{2n}\frac{t^n}{n!} .$$

On the left, we used series property (7). On the right, we multiplied $A(t)\,B(t)$ and used algebraic properties of α and β, and then combined our knowledge of Eqs. (1) through (4). Lastly, equating coefficients of $t^n/n!$ gives us the identity

$$\sum_{k=0}^{n}\binom{n}{k}F_k = F_{2n} .$$

If we follow the same steps with $B(t) = e^{-t}$ and $A(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta)$, then

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_k\right)\frac{t^n}{n!} = \frac{e^{(\alpha-1)t} - e^{(\beta-1)t}}{\alpha - \beta}$$

(9)

$$= \frac{e^{-\beta t} - e^{-\alpha t}}{\alpha - \beta} = \sum_{n=0}^{\infty}(-1)^{n+1}F_n\frac{t^n}{n!} .$$

The identity resulting from (9) is

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_k = (-1)^{n+1}F_n .$$

The technique, then, is this: Take $B(t)$ and $A(t)$ as simple functions in terms of powers of e. Follow algebra as outlined in Eqs. (1) through (7), and equate coefficients of $t^n/n!$ The reader is invited to use $B(t) = e^{-t}$ and $A(t) = (e^{\alpha^2 t} - e^{\beta^2 t})/(\alpha - \beta)$ to derive

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F_{2k} = F_n .$$

For an identity relating Fibonacci and Lucas numbers, let

$$A(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta) , \qquad B(t) = e^{\alpha t} + e^{\beta t} .$$

Since B(t) is the generating function for Lucas number coefficients (see Eq. (6)), computing the series product A(t) B(t) gives

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k L_{n-k}\right)\frac{t^n}{n!} = \frac{e^{2\alpha t} - e^{2\beta t}}{\alpha - \beta} = \sum_{n=0}^{\infty} 2^n F_n \frac{t^n}{n!}\,, \tag{10}$$

yielding

$$\sum_{k=0}^{n}\binom{n}{k}F_k L_{n-k} = 2^n F_n\,.$$

Similarly, let $A(t) = B(t) = (e^{\alpha t} - e^{\beta t})/(\alpha - \beta)$, leading to

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_k F_{n-k}\right)\frac{t^n}{n!} = \left(\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}\right)^2 = \frac{1}{5}\left(e^{2\alpha t} + e^{2\beta t} - 2e^t\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left(2^n L_n - 2\right)\frac{t^n}{n!}\,, \tag{11}$$

$$\sum_{k=0}^{n}\binom{n}{k}F_k F_{n-k} = \frac{1}{5}\left(2^n L_n - 2\right)\,.$$

The reader should use $A(t) = B(t) = e^{\alpha t} + e^{\beta t}$ to derive

$$\sum_{k=0}^{n}\binom{n}{k}L_k L_{n-k} = 2^n L_n + 2\,. \tag{12}$$

To generalize, try combinations using $e^{\alpha^m t}$ and $e^{\beta^m t}$, such as

$$A(t) = \left(e^{\alpha^m t} - e^{\beta^m t}\right)/(\alpha - \beta), \qquad B(t) = e^{\alpha^m t} + e^{\beta^m t}\,,$$

which generalize Eq. (10) as follows:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk} L_{mn-mk}\right)\frac{t^n}{n!} = \frac{e^{2\alpha^m t} - e^{2\beta^m t}}{\alpha - \beta} = \sum_{n=0}^{\infty} 2^n F_{mn}\frac{t^n}{n!}\,. \tag{10'}$$

By taking $A(t) = B(t) = \left(e^{\alpha^m t} - e^{\beta^m t}\right)/(\alpha - \beta)$, Eq. (11) becomes

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{mk}F_{mn-mk}\right)\frac{t^n}{n!} = \left(\frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta}\right)^2$$

(11')
$$= \frac{1}{5}\left(e^{2\alpha^m t} + e^{2\beta^m t} - 2e^{(\alpha^m+\beta^m)t}\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\left(2^n L_{mn} - 2L_m^n\right)\frac{t^n}{n!} \quad .$$

The generalization of (12) found by $A(t) = B(t) = e^{\alpha^m t} + e^{\beta^m t}$ is

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}L_{mk}L_{mn-mk}\right)\frac{t^n}{n!} = \left(e^{\alpha^m t} + e^{\beta^m t}\right)^2$$

(12')
$$= e^{2\alpha^m t} + e^{2\beta^m t} + 2e^{(\alpha^m+\beta^m)t}$$

$$= \sum_{n=0}^{\infty}\left(2^n L_{mn} + 2L_m^n\right)\frac{t^n}{n!}$$

The reader should now experiment with other simple functions involving powers of e. A suggestion is to use some combinations which lead to hyperbolic sines or cosines, which are defined in terms of e.

5. GENERATING FUNCTIONS FOR MORE GENERALIZED IDENTITIES

To get identities of the type

$$\sum_{k=0}^{n}\binom{n}{k}F_{k+r} = F_{2n+r}$$

note that the r^{th} derivative with respect to t of $A(t)$ is

$$D_t^r A(t) = \sum_{n=0}^{\infty} a_{n+r}\frac{t^n}{n!}$$

so that if $A(t) = (e^{\alpha t} + e^{\beta t})/(\alpha - \beta)$, $B(t) = e^t$,

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{k+r}\right)\frac{t^n}{n!} = e^t D_t^r\left(\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta}\right) = \frac{\alpha^r e^{(\alpha+1)t} - \beta^r e^{(\beta+1)t}}{\alpha - \beta}$$

(13)
$$= \frac{\alpha^r e^{\alpha^2 t} - \beta^r e^{\beta^2 t}}{\alpha - \beta} = \sum_{n=0}^{\infty} F_{2n+r}\frac{t^n}{n!}$$

all of which suggests a whole family of identities; e.g., for

$$A(t) = (e^{\alpha^{4m}t} - e^{\beta^{4m}t})/(\alpha - \beta), \qquad B(t) = e^t,$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{4mk+r}\right)\frac{t^n}{n!} = \frac{\alpha^{4rm}e^{(\alpha^{4m}+1)t} - \beta^{4rm}e^{(\beta^{4m}+1)t}}{\alpha - \beta}$$

$$(14) \qquad = \frac{\alpha^{4rm}e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})t} - \beta^{4rm}e^{\alpha^{2m}(\alpha^{2m}+\beta^{2m})t}}{\alpha - \beta}$$

$$= \sum_{n=0}^{\infty} L_{2m}^{n}\,F_{2mn+4mr}\frac{t^n}{n!}\,.$$

From the other direction one can get identities of the type

$$\sum_{n=0}^{\infty} F_{mn}\frac{t^n}{n!} = \frac{e^{\alpha^m t} - e^{\beta^m t}}{\alpha - \beta} = \frac{e^{(\alpha F_m + F_{m-1})t} - e^{(\beta F_m + F_{m-1})t}}{\alpha - \beta}$$

$$(15) \qquad = e^{F_{m-1}t}\left(\frac{e^{\alpha F_m t} - e^{\beta F_m t}}{\alpha - \beta}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{m-1}^{n-k}F_m^k F_k\right)\frac{t^n}{n!}\,.$$

Taking the r^{th} derivative of Eq. (15) leads to

$$(16) \qquad \sum_{n=0}^{\infty} F_{mn+rm}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}F_{m-1}^{n-k}F_m^k F_{k+rm}\right)\frac{t^n}{n!}\,.$$

Replace rm by s in Eq. (16) and compare with Vinson's result [4, p. 38]. See also H. Leonard [5].

REFERENCES

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