ANOTHER PROOF FOR A CONTINUED FRACTION IDENTITY

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Denote the convergents of the continued fraction (Pringsheim's notation [2]),

$$\left[0;\ a_n / b_n\right]_{n=1}$$

by P_n/Q_n, $n = 0, 1, 2, \cdots$, where $P_0/Q_0 = 0/1$. Denote the convergents of the "cut off" continued fraction

$$\left[0;\ a_n / b_n\right]_{n=m+1}$$

by $P_{m,k}/Q_{m,k}$, where $P_{m,o}/Q_{m,o} = 0$, $P_{m,1}/Q_{m,1} = a_{m+1}/b_{m+1}$, etc. Now,

$$P_{m+k} = \left|\begin{array}{cccccc|cccc}
0 & -1 & 0 & 0 & \cdots & & & & & 0 \\
a_1 & b_1 & -1 & 0 & \cdots & & & & & 0 \\
0 & a_2 & b_2 & -1 & \cdots & & & & & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & & a_m & b_m & -1 & 0 & & \\
\hline
 & & & & 0 & a_{m+1} & b_{m+1} & -1 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & & & & & & a_{m+k} & b_{m+k}
\end{array}\right|_{(m+k+1)} .$$

LaPlace's expansion applied to the last k columns gives

$$P_{m+k} = P_m Q_{m,k} - \left|\begin{array}{cccccc}
0 & -1 & & & & \\
a_1 & b_1 & -1 & & & \\
 & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & a_{m-1} & b_{m-1} & -1 \\
 & & & & & a_{m+1}
\end{array}\right| \left|\begin{array}{cccccc}
-1 & & & & & \\
a_{m+2} & b_{m+2} & -1 & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & a_{m+k-1} & b_{m+k-1} & -1 & \\
 & & & a_{m+k} & b_{m+1} &
\end{array}\right|_{(k)}$$

or

$$P_{m+k} = P_m Q_{m,k} + a_{m+1} P_{m-1} \begin{vmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & * & * & \cdots & * \\ 0 & a_{m+2} & b_{m+2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & \cdots\cdots & & b_{m+k} \end{vmatrix}_{(k+1)}$$

where the places denoted by the asterisks may be filled in by any quantities desired. Hence, a_{m+1} is introduced in this last determinant by choosing the second row to be

$$a_{m+1}, \quad b_{m+1}, \quad -1, \quad 0, \quad 0, \quad \cdots, \quad 0$$

and get

$$P_{m+k} = Q_{m,k} P_m + P_{m,k} P_{m-1} \ ,$$

Similarly,

$$Q_{m+k} = Q_{m,k} Q_m + P_{m,k} Q_{m-1} \ .$$

These results may be derived without the use of determinants [1, p. 40] but the procedure is rather lengthy.

REFERENCES

1. Alexey N. Khovanskii, "The Applications of Continued Fractions," translated to English by Peter Wynn, P. Noordhoff, Ltd., Groningen: The Netherlands, 1963.

2. A. Pringsheim, "Ueber die Convergence unendlicher Kettenbruche," Sitzungsber. der Math. Phys. Klasse der Kgl. Bayer. Akad. Wiss., Munchen 28 (1898), pp. 295-324.