ON THE PERIODICITY OF THE TERMINAL DIGITS IN THE FIBONACCI SEQUENCE

DANIEL LANCE HERRICK
Barrington College, Barrington, Rhode Island

In the Fibonacci Quarterly, Vol. 1, No. 2, page 84, Stephen P. Geller reported on a computation (using an IBM 1620) in which he established the period of the Fibonacci numbers modulo 10^n for $n = 1, 2, 3, 4, 5, 6$. For example, the last digit of the decimal numeral for F_k is periodic with period 60, and the last six digits are periodic with period 1,500,000. Mr. Geller closed his report by saying, "There does not yet seem to be any way of guessing the next period," and expresses a hope that a clever computer program could be designed for skipping part of the sequence. And Mr. Geller and R. B. Wallace proposed the finding of an expression for these periods as Problem B15.

In the Quarterly, Vol. 1, No. 4, page 21, Dov Jarden, with all of the scorn of the theoretician for the empiricist, brings out the big guns and batters the problem to pieces, showing that F_k is periodic modulo 10^n with period $15 \cdot 10^{n-1}$ if $n \geq 3$, for $n = 1, 2$ the periods are 60 and 300.

And in the Quarterly, Vol. 2, No. 3, page 211, Richard L. Heimer reported on a calculation examining the same problem in numerals of radix $2, 3, 4, 5, \cdots, 16$. (In his article he does not mention a machine and probably did the calculation by hand.) He writes that his interest was aroused by the eccentricity of the first two periods for decimal numerals.

At the same time as I recently read these articles, I stumbled on the big guns necessary to almost completely reduce the problem, "What is the period of the last j digits of the numeral of radix n of F_k, the k^{th} term in the Fibonacci Sequence?," to a routine computation. (I say almost completely because, for example, $n = 241$ would require extended calculations with large numbers or the use of tables that I don't have available.) The problem is equivalent to:

What is the period of the Fibonacci sequence modulo n^j?

Definition 1. The period of the Fibonacci sequence modulo m, which we write $P(m)$, is the smallest natural number k such that $F_{n+k} \equiv F_n \pmod{m}$ for every natural number n.

We start the subscripts of the Fibonacci sequence in the usual place; that is, $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

All of the theorems necessary to solve this problem have been proven already. We will quote them here as we develop the need for them and close the paper by commenting on where proofs can be found.

Theorem 1. F_k is periodic modulo m for every natural number $m > 1$.

Hence, there is a solution.

To solve the problem for all natural numbers n, it will suffice to solve it for prime numbers, for

Theorem 2. If $m = p_1^{j_1} \cdot p_2^{j_2} \cdot \cdots \cdot p_i^{j_i}$ where the p's are distinct primes, then

$$P(m) = \mathrm{LCM}(P(p_1^{j_1}), \cdots, P(p_i^{j_i}))$$

and we can find the period modulo m if we know the periods of the powers of the prime factors of m. We need one more technical term to talk easily about the problem:

Definition 2. If p is a prime number, the rank of apparition of p, R(p), is the subscript of the first Fibonacci number divisible by p. That is, R(1) is the least natural number k such that p/F_k.

There is a reasonably nice relationship between R(p) and P(p):

Theorem 3. If $p > 2$ is prime,

$$\frac{P(p)}{R(p)} = \begin{cases} 1 & \text{if } R(p) \equiv 2 \pmod 4 \\ 2 & \text{if } R(p) \equiv 0 \pmod 4 \\ 4 & \text{if } R(p) \equiv \pm 1 \pmod 4 \end{cases} .$$

Thus, if we can find the rank of p, we have the period. For many primes, we can find this ratio without knowing the rank of p.

Theorem 4.

$$\frac{P(p)}{R(p)} \begin{cases} = 1 & \text{if } p \equiv 11 \text{ or } 19 \pmod{20} \\ = 2 & \text{if } p \equiv 3 \text{ or } 7 \pmod{20} \\ = 4 & \text{if } p \equiv 13 \text{ or } 17 \pmod{20} \\ \neq 2 & \text{if } p \equiv 21 \text{ or } 29 \pmod{40} \end{cases} .$$

There is a limit to the amount of work involved in finding R(p).

Theorem 5.

$$R(p) \,|\, (p - 1) \quad \text{if } p \equiv \pm 1 \pmod{10}$$

$$R(p) \,|\, (p + 1) \quad \text{if } p \equiv \pm 3 \pmod{10}$$

so that checking somewhat fewer than p/2 Fibonacci numbers is guaranteed to find the first Fibonacci number divisible by p.

Theorem 6. If $P(p^2) \neq P(p)$ then $P(p^j) = p^{j-1}P(p)$.

Thus, subject to a rather odd condition, if we know P(p) we know $P(p^j)$. So far as I know, neither has $P(p^2) \neq P(p)$ been proved nor has a counter-example been found. Just in case, there are theorems to take care of odd situations that might arise:

Theorem 7.

$$\frac{P(p^k)}{R(p^k)} = \frac{P(p)}{R(p)}$$

for prime $p > 2$.

Theorem 8. If t is the largest integer such that $P(p^t) = P(p)$ then

$$P(p^k) = p^{k-t}P(p) \quad \text{for} \quad k > t .$$

Table

t(m) denotes P(m)/R(m); in the last three columns, $n > 2$

m	t(m)	R(m)	P(m)	t(m²)	R(m²)	P(m²)	$t(m^n)$	$R(m^n)$	$P(m^n)$
2	1	3	3	1	6	6	2	$3\cdot 2^{n-2}$	$3\cdot 2^{n-1}$
3	2	4	8	2	12	24	2	$4\cdot 3^{n-1}$	$8\cdot 3^{n-1}$
4		6	6		12	24		$3\cdot 2^{2n-2}$	$3\cdot 2^{2n-1}$
5	4	5	20	4	25	100	4	5^n	$4\cdot 5^n$
6		12	24		12	24		$3\cdot 6^{n-2}$	6^{n-1} *
7	2	8	16	2	56	112	2	$8\cdot 7^{n-1}$	$16\cdot 7^{n-1}$
8		6	12		48	96		$3\cdot 2^{3n-2}$	$3\cdot 2^{3n-1}$
9		12	24		108	216		$4\cdot 3^{2n-1}$	$8\cdot 3^{2n-1}$
10		15	60		150	300		$75\cdot 10^{n-2}$	$15\cdot 10^{n-1}$
11	1	10	10	1	110	110	1	$10\cdot 11^{n-1}$	$10\cdot 11^{n-1}$
12		12	24		12	24		12^{n-1}	$2\cdot 12^{n-1}$
13	4	7	28	4	91	364	4	$7\cdot 13^{n-1}$	$28\cdot 13^{n-1}$
14		24	48		168	336		$21\cdot 14^{n-2}$	$3\cdot 14^{n-1}$ **
15		20	40		300	600		$20\cdot 15^{n-1}$	$40\cdot 15^{n-1}$
16		12	24		192	384		$3\cdot 2^{4n-2}$	$3\cdot 2^{4n-1}$
17	4	9	36	4	153	612	4	$9\cdot 17^{n-1}$	$36\cdot 17^{n-1}$
18		12	24		108	216		$27\cdot 18^{n-2}$	$3\cdot 18^{n-1}$ †
19	1	18	18	1	342	342	1	$18\cdot 19^{n-1}$	$18\cdot 19^{n-1}$
20		30	60		300	600		$15\cdot 20^{n-1}$	$30\cdot 20^{n-1}$
21		8	16		168	336		$8\cdot 21^{n-1}$	$16\cdot 21^{n-1}$
22		30	30		330	330		$165\cdot 22^{n-2}$	$15\cdot 22^{n-1}$
23	2	24	48	2	552	1104	2	$24\cdot 23^{n-1}$	$48\cdot 23^{n-1}$
24		12	24		48	96		$2\cdot 24^{n-1}$	$4\cdot 24^{n-1}$
25		25	100		625	2500		5^{2n}	$4\cdot 5^{2n}$
26		21	84		546	1092		$273\cdot 26^{n-2}$	$21\cdot 26^{n-1}$ ‡
27		36	72		972	1944		$4\cdot 3^{3n-1}$	$8\cdot 3^{3n-1}$
28		24	48		84	168		$3\cdot 28^{n-1}$	$6\cdot 28^{n-1}$
29		14	14	1	406	406	1	$14\cdot 29^{n-1}$	$14\cdot 29^{n-1}$

*holds for $n > 3$; for $n > 2$, $R(6^n) = 3^{n-1}\,\mathrm{LCM}(2^{n-2}, 4)$ and $P(6^n) = 2R(6^n)$

**holds for $n > 4$; for $n > 2$, $R(14^n) = 3\cdot 7^{n-1}\,\mathrm{LCM}(8, 2^{n-2})$, and $P(14^n) = 2R(14^n)$

†holds for $n > 3$; for $n > 2$, $R(18^n) = 3^{2n-1}\,\mathrm{LCM}(4, 2^{n-2})$, and $P(18^n) = 2R(18^n)$

‡R holds for $n > 2$; P holds only for $n > 3$, for $n > 2$

$$P(26^n) = 21 \cdot 13^{n-1}\,\mathrm{LCM}(4,\ 2^{n-1})$$

The original problem can, in principle, be solved for any natural number m by, first, using the fundamental theorem of arithmetic to write m as a product of powers of distinct primes,

$$m = p_1^{j_1} \cdot p_2^{j_2} \cdot \cdots \cdot p_i^{j_i} \;;$$

second, finding $R(p_k)$, $1 \le k \le i$, using Theorem 5 to save labor; third, checking whether $R(p_k^2) = R(p_k)$ and using Theorem 3 or 4, Theorem 7 and Theorem 6 or 8 to find

$$P(p_k^{j_k}) \;;$$

and, finally, using Theorem 2 to find $P(m)$. The same algorithm works for $m = n, n^2, n^3, \ldots$.

After learning these strange things, I constructed a table, starting with $m = 2$ because 2 was the natural place to start and going to 28 because my paper had 27 lines and then adding 29 because it seemed a shame to stop when the next entry would be prime.

We can now shed light on the question that aroused Mr. Heimer — why are the first few periods for decimal numerals irregular? The answer appears when we construct

$$\text{LCM}\,(P(2^k),\ P(5^k)) = \text{LCM}\,(3\cdot 2^{k-1},\ 4\cdot 5^{k-1})$$

in which the exponent of 2 does not start to grow until the 2^2 in $P(5^k)$ is used up. The same thing happens when $m = 18$, for example. See the notes for the table.

I suspect that there is not much more to say about the periodicity of the terminal digits of F_k. The matter of the periodicity of F_k modulo p is an interesting one for labor-saving purposes when one is seeking the prime factorizations of large Fibonacci numbers. In order that this article contain all of the elementary machinery for working on this problem, I quote one more theorem.

Theorem 9. If a is a divisor of F_k then a is a divisor of F_{nk} for every natural number n.

In particular F_k/F_{nk} and p/F_k where k is a multiple of $R(p)$.

Theorems 1 and 2 are theorems 1 and 2, respectively, in Wall. Theorems 3, 4, and 5 are Theorem 2, Theorem 4, and Lemma 3, respectively, in Vinson; Theorems 6 and 8 are Theorem 5 in Wall; Theorem 7 is a Corollary of Vinson's Theorem 2, and Theorem 9 is a Corollary of Theorem 3 in Wall.

REFERENCES

1. John Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," The Fibonacci Quarterly, Vol. 1, No. 2, 1963, p. 37.
2. D. D. Wall, 'Fibonacci Series Modulo m," The American Math. Monthly, Vol. 67, 1960, pp. 525-532.