THE Z TRANSFORM AND THE FIBONACCI SEQUENCE

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Definition. The z transform of f(n) is the function

$$\zeta[f(n)] = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}, \qquad |z| > \frac{1}{\rho}$$

where z is a complex variable and ρ is the radius of convergence of the series.

Applying the z transform to the recursion relation

$$f_{n+2} = f_{n+1} + f_n,$$

we obtain

$$\zeta[f_{n+2}] = \zeta[f_{n+1} + f_n] = \zeta[f_{n+1}] + \zeta[f_n] .$$

Using the shifting theorem for z transforms,

$$\zeta[f(n+m)] = z^m[F(z) - F_m(z)] ,$$

where

$$F_m(z) = \sum_{k=0}^{m-1} f(k)z^{-k} ,$$

which yields

$$z^2[F(z) - F_2(z)] = z[F(z) - F_1(z)] + F(z)$$

and

$$(z^2 - z - 1)F(z) = z^2F_2(z) - zF_1(z) .$$

Hence

$$F(z) = \frac{z^2[f(0) - f(1)z^{-1}] - z[f(0)]}{z^2 - z - 1} ,$$

where

$$z^2 - z - 1 \neq 0 .$$

Since $f_0 = 0$ and $f_1 = 1$, we have

$$F(z) = \frac{z}{z^2 - z - 1} .$$

$F(z)$ is a Laurent series. Therefore, we can multiply $F(z)$ by z^{n-1} and integrate it around a circle for which $|z| > R$. This gives

$$\int_\Gamma F(z)z^{n-1}dz = 2\pi i f(n)$$

or

$$f(n) = \frac{1}{2\pi i}\int_\Gamma F(z)z^{n-1}dz = \Sigma \text{ Residues of } F(z)z^{n-1} .$$

Hence

$$f(n) = \Sigma \text{ Residues}\left[\frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)}\right]z^{n-1}$$

$$= \lim_{z\to\frac{1+\sqrt{5}}{2}}\left[\frac{z^n}{z - \frac{1-\sqrt{5}}{2}}\right] + \lim_{z\to\frac{1-\sqrt{5}}{2}}\left[\frac{z^n}{z - \frac{1+\sqrt{5}}{2}}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^n \Big/ \sqrt{5} - \left(\frac{1-\sqrt{5}}{2}\right)^n \Big/ \sqrt{5} .$$

Therefore

$$f(n) = (\alpha^n - \beta^n)/\sqrt{5} ,$$

where

$$\alpha^n = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

and

$$\beta^n = \left(\frac{1-\sqrt{5}}{2}\right)^n ,$$

which is Binet's formula.