

SOME GENERAL FIBONACCI SHIFT FORMULAE

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The reader is probably aware of such formulae as: [1]

$$(a) \quad F_n = F_{k+1}F_{n-k} + F_k F_{n-k-1}$$

$$(b) \quad F_n = (-1)^{k-1} (F_k F_{n+k+1} - F_{k+1} F_{n+k}) .$$

The object of this paper is to prove more general shift formulae. For this purpose, the following notation will be used:

$$F_n F_m = (n:m), \quad F_n = (n:1) = (1:n), \quad \text{etc.}$$

Let

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2} ,$$

then $\alpha\beta = -1$. Now,

$$F_n = (n:1) = \frac{\alpha^n - \beta^n}{\sqrt{5}} .$$

So

$$(n:m) = \frac{1}{5} (\alpha^{n+m} + \beta^{n+m} - \alpha^n \beta^m - \alpha^m \beta^n) .$$

Replace n by x , and m by $n + m - x$. Then,

$$(x:n + m - x) = \frac{1}{5} (\alpha^{n+m} + \beta^{n+m} - \alpha^x \beta^{n+m-x} - \alpha^{n+m-x} \beta^x)$$

$$(n:m) - (x:n + m - x) = \frac{1}{5} (\alpha^x \beta^{n+m-x} + \alpha^{n+m-x} \beta^x - \alpha^n \beta^m - \alpha^m \beta^n)$$

$$\begin{aligned} (c) \quad &= \frac{(\alpha\beta)^m}{5} (\alpha^{x-m} \beta^{n-x} + \alpha^{n-x} \beta^{x-m} - \alpha^{n-m} - \beta^{n-m}) \\ &= \frac{(-1)^{m+1}}{5} (\alpha^{n-m} + \beta^{n-m} - \alpha^{x-m} \beta^{n-x} - \alpha^{n-x} \beta^{x-m}) \\ &= (-1)^{m+1} (x - m:n - x) . \end{aligned}$$

If x is replaced by $-x$, we get

$$(d) \quad (n:m) = (-x:n + m + x) + (-1)^{m+1}(-x - m:n + x).$$

Equations (c) and (d) may be combined into the one formula:

$$(1) \quad (n:m) = (\pm x:n + m \mp x) + (-1)^{m+1}(\pm x - m:n \mp x).$$

By the same method, the following formulae may be proved:

$$(2) \quad (n:m) = (\pm x + m:n \mp x) + (-1)^{m+1}(\pm x:n - m \mp x)$$

$$(3) \quad (n \pm x:m) = (\pm x:n + m) + (-1)^{m+1}(\pm x - m:n)$$

$$(4) \quad (n \pm x:m) = (\pm x + m:n) + (-1)^{m+1}(\pm x:n - m)$$

$$(5) \quad (n[x \pm 1]:m) = (\pm n:nx + m) + (-1)^{m+1}(\pm n - m:nx)$$

$$(6) \quad (n[x \pm 1]:m) = (\pm n + m:nx) + (-1)^{m+1}(\pm n:nx - m)$$

$$(7) \quad (n:m) = (-1)^{x+1}[(x:n + m + x) - (m + x:n + x)]$$

Clearly, Equations (a) and (b) are special cases of Equations (2) and (7), respectively.

REFERENCE

1. Brother Alfred Brousseau, An Introduction to Fibonacci Discovery, P. 46, page 11, and P. 48, page 12.



ERRATA

Please make the following corrections on "A Generalized Fibonacci Numeration," by E. Zeckendorf, appearing on pp. 365-372 of the October, 1972 Fibonacci Quarterly:

p. 366, line 15: Please change the third word from: sequencex to sequences.

p. 368, line 13: Read: $t_{6,3,-1,-4,-6} = F_5 + F_2 + F_{-2} + F_{-5} + F_{-7}$.

line 8 from bottom: Underscore: symmetric pairs.

line 6 from bottom: Read: metric pairs may join up into one symmetric group (e. g., $t_{6,0,-6}$, $t_{8,4,-4,-8}$).

line 4 from bottom: Underscore: saturated symmetric groups.

