ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-230 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pa.

(a) If 5 is a quadratic nonresidue of a prime p ($p \neq 5$), then $p \mid F_{k(p+1)}$, k a positive integer.

(b) If 5 is a quadratic residue of a prime p, then $p \mid F_{k(p-1)}$, k a positive integer.

H-231 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

1. Let $A_0 = 0$, $A_1 = 1$,

$$\begin{cases} A_{2k+1} = A_{2k} + A_{2k-1} \\ A_{2k+2} = A_{2k+1} - A_{2k} \end{cases}.$$

Find A_n.

2. Let $B_0 = 2$, $B_1 = 3$,

$$\begin{cases} B_{2k+1} = B_{2k} + B_{2k-1} \\ B_{2k+2} = B_{2k+1} - B_{2k} \end{cases}.$$

Find B_n.

H-232 Proposed by R. Garfield, the College of Insurance, New York, New York.

Define a sequence of polynomials, $\{G_k(x)\}_{k=0}^{\infty}$ as follows:

$$\frac{1}{1 - (x^2 + 1)t^2 - xt^3} = \sum_{k=0}^{\infty} G_k(x)\, t^k .$$

1. Find a recursion formula for $G_k(x)$.
2. Find $G_k(1)$ in terms of the Fibonacci numbers.
3. Show that when $x = 1$, the sum of any 4 consecutive G numbers is a Lucas number.

H-233 Proposed by A. G. Shannon, NSW Institute of Technology, Broadway, and The University of New England, Armidale, Australia.

The notation of Carlitz* suggests the following generalization of Fibonacci numbers. Define

$$f_n^{(r)} = (a^{nk+k} - b^{nk+k})/(a^k - b^k),$$

where $k = r - 1$, and a, b are the zeros of $x^2 - x - 1$, the auxiliary polynomial of the ordinary Fibonacci numbers, $f_n^{(2)}$.

Show that

$$\text{(a)} \qquad \sum_{n=0}^{\infty} f_n^{(r)} x^n = 1/(1 - (a^k + b^k)x + (a^k b^k)x^2).$$

Let $f_k = (a^{k+1} - b^{k+1})/(a - b)$, and prove that

$$\text{(b)} \qquad f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}.$$

(Note that when $r = 2$ (and so $k = 1$), $f_k = f_{k-1} = 1$, $f_{k-2} = 0$, and (b) reduces to the well known

$$f_n^{(2)} = \sum_{0 \le 2m \le n} \binom{n-m}{m}. \quad)$$

SOLUTIONS

SUCCESS!

Editorial Note. We previously listed H-61, H-73, and H-77 as unsolved. However, this is incorrect. H-61 is solved in Vol. 5, No. 1, pp. 72-73. H-73 is solved in Vol. 5, No. 3, pp. 255-256. H-77 is solved in Vol. 5, No. 3, pp. 256-258.

ANOTHER OLDIE

H-62 Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.

Find all polynomials $f(x)$ and $g(x)$, of the form

$$f(x + 1) = \sum_{j=0}^{r} a_j x^j, \qquad a_j \text{ an integer},$$

*L. Carlitz, "The Characteristic Polynomial of a Certain Matrix of Binomial Coefficients," Fibonacci Quarterly, Vol. 3 (1965), pp. 81-89.

$$g(x) = \sum_{j=0}^{s} b_j x^j, \qquad b_j \text{ an integer},$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\}$$
$$+ 2(x+1)\left\{xf(x+1) - (x+1)g(x)\right\} = 0\,.$$

H-87 Proposed by Monte Boisen, Jr., San Jose State University, San Jose, Calif.

Show that, if

$$u_0 = u_2 = u_3 = \cdots = u_{n-1} = 1$$

and

$$u_k = u_{k-1} + u_{k-2} + \cdots + u_{k-n} \qquad k \ge n\,,$$

then

$$\frac{1 - x^2 - 2x^3 - \cdots - (n-2)x^{n-1}}{1 - x - x^2 - \cdots - x^n} = \sum_{k=0}^{\infty} u_k x^k\,.$$

Solution by Clyde A. Bridger, Springfield, Illinois.

Write

$$g(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}\,,$$
$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and

$$g/f = q(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n + A_{n+1} x^{n+1} + \cdots\,,$$

where a_0 and a_n are not zero, at least one c_i is not zero, and $f(x) = 0$ has no multiple roots.

Then g/f generates a recurrence of length n, as is at once apparent by either long division or by equating coefficients of like powers of x in $g(x) = f(x)\cdot q(x)$. The first n A's depend entirely on the c's, as the following set of equations shows.

$$c_0 = a_0 A_0$$
$$c_1 = a_0 A_1 + a_1 A_0$$
$$c_2 = a_0 A_2 + a_1 A_1 + a_2 A_0$$
$$\cdots\cdots\cdots$$
$$c_{n-1} = a_0 A_{n-1} + a_1 A_{n-2} + \cdots + a_{n-1} A_0$$
$$0 = a_0 A_n + a_1 A_{n-1} + \cdots + a_n A_0$$
$$\cdots\cdots\cdots$$
$$0 = a_0 A_k + a_1 A_{k-1} + \cdots + a_k A_0 \qquad (k \ge n)$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$

Beginning with the $(n+1)^{st}$ equation, A_n can be expressed directly in terms of the $n-1$ preceding A's. The general term A_k for any $k \geq n$ provides the formula or difference equation that the fraction g/f generates.

To solve the given problem, one sets $c_0 = 1$ and $c_i = -(n-1)$ for $i = 1$ to $i = n - 1$, $a_0 = 1$, $a_i = -1$ for $i = 1$ to n, and $A_i = u_i$.

FIT TO A "T"

H-197 Proposed by Lawrence Somer, University of Illinois, Urbana, Illinois.

Let

$$\left\{ u_n^{(t)} \right\}_{n=1}^{\infty}$$

be the t-Fibonacci sequences with positive entries satisfying the recursion relationship:

$$u_n^{(t)} = \sum_{i=1}^{t} u_{n-i} .$$

Find

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \frac{u_{n+1}^{(t)}}{u_n^{(t)}} .$$

Solution by the Proposer.

By analyzing the convergents of continued fractions, one can easily see that for any fixed t, and any initial entries, $u_1^{(t)}, u_2^{(t)}, \cdots, u_t^{(t)}$,

$$\lim_{n \to \infty} \frac{u_{n+1}^{(t)}}{u_n^{(t)}}$$

will be a constant. Let

$$u_1^{(t)} = 1, \quad u_2^{(t)} = 2, \quad \cdots, \quad u_t^{(t)} = 2^t .$$

For this choice we have

$$u_{t+1}^{(t)} = 2^{t+1} - 1 \quad \text{and} \quad u_{t+2}^{(t)} = 2^{t+2} - 3 .$$

Let

$$\psi = \lim_{n \to \infty} \frac{u_{n+1}^{(t)}}{u_n^{(t)}} .$$

If one examines the convergents of the continued fractions, he finds

$$\frac{u_{t+1}^{(t)}}{u_t^{(t)}} < \psi < \frac{u_{t+2}^{(t)}}{u_{t+1}^{(t)}}$$

for large enough n. Thus

$$\frac{2^{t+1} - 1}{2^t} < \psi < \frac{2^{t+2} - 3}{2^{t+1} - 1}$$

or

$$2 - \frac{1}{2^t} < \psi < 2 - \frac{1}{2^{t+1} - 1}$$

and

$$\lim_{t \to \infty} \psi = 2 .$$

It thus follows that the desired limit is 2.

Also solved by P. Tracy and one unsigned solver.

PELL-MELL

H-198 Proposed by E. M. Cohn, National Aeronautics and Space Administration, Washington, D.C.

There is an infinite sequence of square values for triangular numbers,*

$$k^2 = m(m + 1)/2 .$$

Find simple expressions for k and m in terms of Pell numbers, P_n ($P_{n+2} = 2P_{n+1} + P_n$, where $P_0 = 0$ and $P_1 = 1$).

Solution by the Proposer.

Since $(m, m + 1) = 1$, the odd factor must be a square, say $(2s + 1)^2$. Then the even factor is $(2s^2 + 2s)$ or $(2s^2 + 2s + 1)$. (After division by 2.)

Let $k^2/(2s + 1)^2 = q^2$, so that either

$$2s^2 + 2s - q^2 = 0$$

or

$$2s^2 + 2s + 1 - q^2 = 0 .$$

Solving for s and re-arranging,

$$(2s + 1)^2 = 2q^2 \pm 1$$

$$k = q\sqrt{2q^2 \pm 1} .$$

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*A. W. Sylvester, Amer. Math. Monthly, 69 (1962), p. 168.

It has been shown* that $q_n = P_n$ for Diophantine solutions of such discriminants, and that the discriminant itself equals $P_{n+1} - P_n$. Furthermore, $P_n(P_{n+1} - P_n) = \frac{1}{2} P_{2n}$. Thus

$$k_n = \tfrac{1}{2} P_{2n} \quad .$$

For even n,

$$m_n = 2 P_n^2 \ ,$$

and for odd n,

$$m_n = 2 P_n^2 \pm 1 \ .$$

Since even Pell numbers are alternately congruent to 0 (mod 4) and 2 (mod 4), pairs of values of k are of different parity.

Also solved by P. Bruckman, who also solved H-192, H-193, and H-194.

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*E. M. Cohn, submitted to the Fibonacci Quarterly.

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