FIBONACCI NOTES
1. ZERO-ONE SEQUENCES AND FIBONACCI NUMBERS OF HIGHER ORDER

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1. INTRODUCTION

It is well known (see, for example [1, p. 14]) that the number of sequences of zeros and ones of length n with two consecutive ones forbidden is equal to F_{n+2}. For example for $n = 3$, the allowable sequences are

$$(000), \quad (100), \quad (010), \quad (001), \quad (101) .$$

As a first extension of this result, let $f(n,k)$ denote the number of sequences of length n.

$$(a_1, a_2, \cdots, a_n) \qquad (a_i = 0 \text{ or } 1) \tag{1.1}$$

such that

$$a_i a_{i+1} \cdots a_{i+k} = 0 \qquad (i = 1, 2, \cdots, n - k) ; \tag{1.2}$$

that is, a string of $k + 1$ consecutive ones is forbidden. Also, let $f(n,k,r)$ denote the number of such sequences with exactly r ones and let $f_j(n,k,r)$ denote the number of such sequences with r ones and beginning with j ones, $0 \le j \le r$.

It suffices to evaluate $f_0(n,k,r)$. We shall show that

$$f_0(n,k,r) = c_k(n - r, r) , \tag{1.3}$$

where $c_k(n,r)$ is defined by

$$(1 + x + \cdots + x^k)^n = \sum_{r=0}^{kn} c_k(n,r)\, x^r . \tag{1.4}$$

Also if

$$f_0(n,k) = \sum_r f_0(n,k,r)$$

is the number of allowable sequences beginning with at least one zero, we have

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$$(1.5) \qquad \sum_{n=0}^{\infty} f_0(n,k)\,x^n = \frac{1}{1 - x - x^2 - \cdots - x^{k+1}} .$$

In the next place let r, s be fixed positive integers and let $f(m,n;\, r,s)$ denote the number of sequences of length $m + n$

$$(a_1,\ a_2,\ \cdots,\ a_{m+n}) \qquad (a_i = 0 \text{ or } 1)$$

with exactly m zeros and n ones, at most r consecutive zeros and at most s consecutive ones. Also, let $f_j(m,n;\, r,s)$ denote the number of such sequences beginning with exactly j zeros, $0 \le j \le r$; let $\overline{f}_k(m,n;\, r,s)$ denote the number of such sequences beginning with exactly k ones, $0 \le k \le s$.

As in the previous problem, it suffices to evaluate $f_0(m,n;\, j,k)$ and $\overline{f}_0(m,n;\, j,k)$. We shall show that

$$(1.6) \qquad f_0(m,n;\, r,s) = \sum_k c_{r-1}(k,\ m-k)c_{s-1}(k,\ n-k) + \sum_k c_{r-1}(k+1,\ m-k-1)c_{s-1}(k,\ n-k) ,$$

$$(1.7) \qquad \overline{f}_0(m,n;\, r,s) = \sum_k c_{r-1}(k,\ m-1)c_{s-1}(k,\ n-k) + \sum_k c_{r-1}(k,\ m-k)c_{s-1}(k+1,\ n-k-1) ,$$

$$(1.8) \qquad f(m,n;\, r,s) = \sum_k \left\{c_{r-1}(k,\ m-k)c_{s-1}(k,\ n-k) + c_{r-1}(k+1,\ m-k-1)c_{s-1}(k,\ n-k) + c_{r-1}(k,\ m-k)c_{s-1}(k+1,\ n-k-1)\right\} .$$

As a further extension one can consider sequences of 0's, 1's, and 2's, say with restrictions on the number of allowable consecutive elements of each kind. However, we leave this for another occasion.

2. FIRST PROBLEM

We now consider the first problem as defined above. It is clear from the definition that

$$f(n,k,r) = \sum_{j=0}^{k} f_j(n,k,r) \quad . \tag{2.1}$$

Also

$$f_0(n,k,r) = \sum_{j=0}^{k} f_j(n-1,k,r) = f(n-1,k,r) \tag{2.2}$$

and

$$f_j(n,k,r) = f_0(n-j,k,r-j) \qquad (0 \le j \le k) . \tag{2.3}$$

Hence $f_0(n,k,r)$ satisfies the mixed recurrence

$$f_0(n,k,r) = \sum_{j=0}^{k} f_0(n-j-1,k,r-j) \qquad (n > k+1) . \tag{2.4}$$

Now, for $r \le k$,

$$f_0(1,k,r) = \delta_{0,r} ,$$

$$f_0(2,k,r) = \begin{cases} 1 & (r = 0,1) \\ 0 & (r > 1) \end{cases} ,$$

$$f_0(3,k,r) = \begin{cases} 1 & (r = 0) \\ 2 & (r = 1) \\ 1 & (r = 2) \\ 0 & (r > 2) \end{cases} .$$

Generally, for $1 \le m \le k+1$, we have

$$f_0(m,k,r) = \begin{cases} \binom{m-1}{r} & (0 \le r < m) \\ 0 & (r \ge m) \end{cases} . \tag{2.5}$$

If we take $n = k+1$, $r \le k$ in (2.4) we get

$$f_0(k+1,k,r) = \sum_{j=0}^{k} f_0(k-j,k,r-j) .$$

By (2.5) this reduces to

$$\binom{k}{r} = \sum_{j=0}^{k-1} \binom{k-j-1}{r-j} + f_0(0,\ k,\ r-k)\ .$$

Since

$$\sum_{j=0}^{k-1} \binom{k-j-1}{r-j} = \sum_{j=0}^{k-1} \binom{k-j-1}{k-r-1} = \sum_{j=0}^{k-1} \binom{j}{k-r-1} = \binom{k}{k-r} = \binom{k}{r},$$

it follows that (2.4) holds for $n \geq k+1 > r$ provided we define

$$f_0(0,k,t) = 0 \qquad (t \leq 0)\ .$$

Moreover (2.4) holds for $r > k$, $n = k+1$, since both sides vanish.

Now put

$$F(x,y) = \sum_{n,r=0}^{\infty} f_0(n,k,r)\, x^n y^r \ . \tag{2.6}$$

Then, by (2.4) and (2.5),

$$\begin{aligned} F(x,y) &= \sum_{n=0}^{k} \sum_{r=0}^{\infty} f_0(n,k,r)\, x^n y^r + \sum_{n=k+1}^{\infty} \sum_{r=0}^{\infty} f_0(n,k,r)\, x^n y^r \\ &= 1 + \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} \binom{n-1}{r} x^n y^r \\ &\quad + \sum_{n=k+1}^{\infty} \sum_{r=0}^{\infty} \sum_{j=0}^{k} f_0(n-j-1,\ k,\ r-j)\, x^n y^r \\ &= 1 + \sum_{n=1}^{k} \sum_{r=0}^{\infty} \binom{n-1}{r} x^n y^r \\ &\quad + \sum_{j=0}^{k} x^{j+1} y^j \sum_{n=k+1}^{\infty} \sum_{r=0}^{\infty} f_0(n-j-1,\ k,\ r-j)\, x^{n-j-1} y^{r-j} \\ &= 1 + \sum_{n=1}^{k} \sum_{r=0}^{\infty} \binom{n-1}{r} x^n y^r \\ &\quad + \sum_{j=0}^{k} x^{j+1} y^j \left\{ F(x,y) = \sum_{n=0}^{k-j-1} \sum_{r=0}^{\infty} f(n,k,r)\, x^n y^r \right\} \ . \end{aligned}$$

Since

$$\sum_{j=0}^{k} x^{j+1} y^j \sum_{n=0}^{k-j-1} \sum_{r=0}^{\infty} f(n,k,r)\, x^n y^r = \sum_{j=0}^{k} x^{j+1} y^j \sum_{n=0}^{k-j-1} \sum_{r=0}^{\infty} \binom{n-1}{r} x^n y^r$$

$$= \sum_{n=1}^{k} \sum_{r} x^n y^r \sum_{j} \binom{n-j-2}{r-j}$$

$$= \sum_{n=1}^{k} \sum_{r} \binom{n-1}{r} x^n y^r \;,$$

it follows that

$$F(x,y) = 1 + \sum_{j=0}^{k} x^{j+1} y^j \cdot F(x,y) \;.$$

Therefore

$$F(x,y) = \frac{1}{1 - x \sum_{j=0}^{k} x^j y^j} \;. \tag{2.7}$$

If we define the coefficient $c_k(n,r)$ by means of

$$(1 + x + \cdots + x^k)^n = \sum_{r=0}^{kn} c_k(n,k)\, x^r \;, \tag{2.8}$$

it is clear that

$$\frac{1}{1 - x \sum_{j=0}^{k} x^y y^j} = \sum_{s=0}^{\infty} x^s \left(\sum_{j=0}^{k} x^j y^j \right)^s \;.$$

$$= \sum_{s=0}^{\infty} x^s \sum_{r=0}^{ks} c_k(s,r)\, x^r y^r$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} c_k(n-r,\, r)\, x^n y^r \;.$$

Therefore by (2.6) and (2.7) we have

(2.9) $$f_0(n,k,r) = c_k(n - r, r)$$

and, in view of (2.2),

(2.10) $$f(n,k,r) = c_k(n - r + 1, r) \ .$$

Moreover if we put

$$f_0(n,k) = \sum_r f_0(n,k,r) ,$$

it follows that

(2.11) $$\sum_{n=0}^{\infty} f_0(n,k)\, x^n = \frac{1}{1 - \sum_{j=1}^{k+1} x^j} \ .$$

In particular, for $k = 1$, it is clear from (2.11) that

(2.12) $$f_0(n,1) = F_{n+1} \ .$$

3. SECOND PROBLEM

We turn now to the second problem defined in the Introduction. It is convenient to define

(3.1) $$\begin{cases} f_0(0,0;\ r,s) = \overline{f}(0,0;\ r,s) = 1 \\ f_j(0,0;\ r,s) = \overline{f}_k(0,0;\ r,s) = 0 \qquad (j > 0,\ k > 0) \ . \end{cases}$$

The following relations follow from the definition.

(3.2) $$\begin{cases} f_0(m,n;\ r,s) = \sum_{k=1}^{s} \overline{f}_k(m,n;\ r,s) \\ \overline{f}_0(m,n;\ r,s) = \sum_{j=1}^{r} f_j(m,n;\ r,s) \end{cases}$$

(3.3) $$\begin{cases} f_j(m,n;\ r,s) = \sum_{k=1}^{s} \overline{f}_k(m - j,\ n;\ r,s) \qquad (j \geq 0) \\ \overline{f}_k(m,n;\ r,s) = \sum_{j=1}^{r} f_j(m,\ n - k;\ r,s) \qquad (k \geq 0) , \end{cases}$$

where it is understood that

$$f_0(m,n;\ r,s) = \overline{f}_0(m,n;\ r,s) = 0$$

if either m or n is negative.

We have also

$$(3.4)\qquad \begin{cases} f_j(m,n;\ r,s) = f_0(m-j,\ n;\ r,s) & (j > 0) \\ \overline{f}_k(m,n;\ r,s) = \overline{f}_0(m,n-k;\ r,s) & (k > 0) \end{cases}.$$

In particular

$$(3.5)\qquad \begin{cases} f_j(j,\ n;\ r,s) = f_0(0,\ n;\ r,s) = 1 & (0 \le n \le s) \\ \overline{f}_k(m,k;\ r,s) = \overline{f}_0(m,0;\ r,s) = 1 & (0 \le m \le r) \end{cases}.$$

It follows from (3.2) and (3.4) that

$$(3.6)\qquad \begin{cases} f_0(m,n;\ r,s) = \sum_{k=1}^{s} \overline{f}_0(m,\ n-k;\ r,\ s) \\ \overline{f}_0(m,n;\ r,s) = \sum_{j=1}^{r} f_0(m-j,\ n;\ r,\ s) \end{cases}$$

and therefore

$$(3.7)\qquad \begin{cases} f_0(m,n;\ r,s) = \sum_{j=1}^{r}\sum_{k=1}^{s} f_0(m-j,\ n-k;\ r,\ s) \\ \overline{f}_0(m,n;\ r,s) = \sum_{j=1}^{r}\sum_{k=1}^{s} \overline{f}_0(m-j,\ n-k;\ r,\ s) \end{cases}.$$

Now put

$$\begin{aligned} F_0 &= F_0(x,y) = \sum_{m,n=0}^{\infty} f_0(m,n;\ r,s)\,x^m y^n \\ \overline{F}_0 &= \overline{F}_0(x,y) = \sum_{m,n=0}^{\infty} \overline{f}_0(m,n;\ r,s)\,x^m y^n \end{aligned}.$$

Then by (3.5) and (3.7),

$$F_0 = \sum_{k=0}^{s} y^k + \sum_{m,n=1}^{\infty} f_0(m,n;\ r,s)\, x^m y^n$$

$$= \sum_{k=0}^{s} y^k + \sum_{m,n=1}^{\infty} \sum_{j=1}^{r} \sum_{k=1}^{s} f_0(m-j,\ n-k;\ r,\ s)\, x^m y^n$$

$$= \sum_{k=0}^{s} y^k + \sum_{j=1}^{r} \sum_{k=1}^{s} x^j y^k \sum_{m,n=0}^{\infty} f_0(m,n;\ r,s)\, x^m y^n$$

$$= \sum_{k=0}^{s} y^k + \sum_{j=1}^{r} \sum_{k=1}^{s} x^j y^k \cdot F_0 \ ,$$

so that

(3.8)
$$F_0 = \frac{\sum_{k=0}^{s} y^k}{1 - \sum_{j=1}^{r} \sum_{k=1}^{s} x^j y^k} \ .$$

Similarly

(3.9)
$$\overline{F}_0 = \frac{\sum_{j=0}^{r} y^j}{1 - \sum_{j=1}^{r} \sum_{k=1}^{s} x^j y^k} \ .$$

Clearly

$$f(m,n;\ r,s) = f_0(m,n;\ r,s) + \overline{f}_0(m,n;\ r,s) \qquad (m+n > 0) \ ,$$

so that

$$F(x,y) = \sum_{m,n=0}^{\infty} f(m,n;\ r,s)\, x^m y^n = -1 + F_0(x,y) + \overline{F}_0(x,y) \ .$$

Hence

(3.10)
$$F(x,y) = \frac{\sum_{j=0}^{r} x^j + \sum_{k=0}^{s} y^k}{1 - \sum_{j=1}^{r} \sum_{k=1}^{s} x^j y^k} \ .$$

To get explicit formulas we take

$$\left(1-\sum_{j=1}^{r}\sum_{k=1}^{s}x^jy^k\right)^{-1} = \sum_{k=0}^{\infty}x^ky^k(1+\cdots+x^{r-1})^k(1+\cdots+y^{s-1})^k$$

$$= \sum_{k=0}^{\infty}x^ky^k\sum_{i=0}^{\infty}c_{r-1}(k,i)x^i\sum_{j=0}^{\infty}c_{s-1}(k,j)x^j$$

$$= \sum_{m,n=0}^{\infty}x^my^n\sum_{k}c_{r-1}(k,\ m-k)c_{s-1}(k,\ n-k)\ .$$

Then

$$\frac{\sum_{j=1}^{r}x^j}{1-\sum_{j=1}^{r}\sum_{k=1}^{s}x^jy^k} = \sum_{k=0}^{\infty}x^{k+1}y^k(1+\cdots+x^{r-1})^{k+1}(1+\cdots+y^{s-1})^k$$

$$= \sum_{k=0}^{\infty}x^{k+1}y^k\sum_{i,j}c_{r-1}(k+1,\ i)c_{s-1}(k,j)x^iy^j$$

$$= \sum_{m,n=0}^{\infty}x^my^n\sum_{k}c_{r-1}(k+1,\ m-k-1)c_{s-1}(k,\ n-k)\ ,$$

$$\frac{\sum_{j=1}^{s}y^k}{1-\sum_{j=1}^{r}\sum_{k=1}^{s}x^jy^k} = \sum_{m,n=0}^{\infty}x^my^n\sum_{k}c_{r-1}(k,\ m-k)c_{s-1}(k+1,\ n-k-1)\ .$$

It follows that

$$(3.11)\qquad \left\{\begin{aligned} f_0(m,n;\ r,s) &= \sum_{k}\{c_{r-1}(k,\ m-k)c_{s-1}(k,\ n-k) \\ &\quad + c_{r-1}(k+1,\ m-k-1)c_{s-1}(k,\ n-k)\}\ , \end{aligned}\right.$$

$$(3.12)\qquad \left\{\begin{aligned} \overline{f}_0(m,n;\ r,s) &= \sum_{k}\{c_{r-1}(k,\ m-k)c_{s-1}(k,\ n-k) \\ &\quad + c_{r-1}(k,\ m-k)c_{s-1}(k+1,\ n-k+1)\}\ , \end{aligned}\right.$$

$$f(m,n;\,r,s) = \sum_k \{ c_{r-1}(k,\ m-k)c_{s-1}(k,\ n-k)$$

(3.13)

$$+ c_{r-1}(k+1,\ m-k-1)c_{s-1}(k,\ n-k)$$
$$+ c_{r-1}(k,\ m-k)c_{s-1}(k+1,\ n-k-1)\} .$$

When $r \to \infty$ the restrictions in the definition of f, f_j reduce to the single restriction that the number of consecutive ones is at most s. The generating functions (3.8), (3.9), (3.10) reduce to

(3.8′)
$$F_0 = \frac{(1-x)\sum_{k=0}^{s} y^k}{1 - x\sum_{k=0}^{s} y^k} ,$$

(3.9′)
$$F_0 = \frac{1}{1 - x\sum_{k=0}^{s} y^k} ,$$

(3.10′)
$$F = \frac{\sum_{k=0}^{s} y^k}{1 - x\sum_{k=0}^{s} y^k} .$$

It can be verified that these results are in agreement with the results of Sec. 2 above.

REFERENCE

1. J. Riordan, An Introduction to Combinatorial Analysis, Wiley New York, 1958.