COMBINATORIAL ANALYSIS AND FIBONACCI NUMBERS

GEORGE E. ANDREWS
The Pennsylvania State University, University Park, Pa. 16802

1. INTRODUCTION

The object of this paper is to present a new combinatorial interpretation of the Fibonacci numbers.

There are many known combinatorial interpretations of the Fibonacci numbers (e.g., [9]); indeed, the original use of these numbers was that of solving the rabbit breeding problem of Fibonacci [10]. The appeal of this new interpretation lies in the fact that it provides combinatorial proofs of several well known Fibonacci identities. Among them:

$$\sum_{j=0}^{n} \binom{n}{j} F_j = F_{2n} .$$

These results will be presented in Section 2. In Section 3, we shall describe further possibilities for exploration of Fibonacci numbers via combinatorics.

2. FIBONACCI SETS

Definition 1. We say a finite set S of positive integers is Fibonacci if each element of the set is $\geq |S|$, where $|S|$ denotes the cardinality of S.

Definition 2. We say a finite set S of positive integers is r-Fibonacci if each element of the set is $\geq |S| + r$.

We note that "0-Fibonacci" means "Fibonacci."

Table 1
Subsets of $\{1, 2, \cdots, n\}$ that are r-Fibonacci

n	Fibonacci	1-Fibonacci	2-Fibonacci
1	ϕ, $\{1\}$	ϕ	ϕ
2	ϕ, $\{1\}$, $\{2\}$	ϕ, $\{2\}$	ϕ
3	ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{3,2\}$	ϕ, $\{2\}$, $\{3\}$	ϕ, $\{3\}$
4	ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$	ϕ, $\{2\}$, $\{3\}$, $\{4\}$, $\{3,4\}$	ϕ, $\{3\}$, $\{4\}$

* Partially supported by National Science Foundation Grant GP-23774.

Theorem 1. There are exactly F_{n+2-r} subsets of $\{1, 2, \cdots, n\}$ that are r-Fibonacci for $n \geq r - 1$.

Proof. When $n = r - 1$ or r, ϕ is the only subset of $\{1, 2, \cdots, n\}$ that is r-Fibonacci, since each element of an r-Fibonacci set must be $>r$. Since $F_1 = F_2 = 1$, we see that the theorem is true for $n = r - 1$ or r.

Assume the theorem true for each n with $r < n \leq n_0$ (and for all r). Let us consider the r-Fibonacci subsets of $\{1, 2, \cdots, n_0, n_0 + 1\}$ that: (1) do not contain $n_0 + 1$, and (2) do contain $n_0 + 1$. Clearly there are F_{n_0+2-r} elements of the first class. If we delete $n_0 + 1$ from each set in the second class, we see that we have established a one-to-one correspondence between the elements of the second class and the $(r + 1)$-Fibonacci subsets of $\{1, 2, \cdots, n_0\}$, hence there are $F_{n_0+2-(r+1)}$ elements of the second class. This means that there are

$$\begin{aligned} &F_{n_0+2-r} + F_{n_0+2-(r+1)} \\ &\quad = F_{(n_0+1)+2-r} \end{aligned}$$

r-Fibonacci subsets of $\{1, 2, \cdots, n_0 + 1\}$, and this completes Theorem 1.

Theorem 2. For $n \geq 0$,

$$\begin{aligned} F_{n+2} &= 1 + \binom{n}{j} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots \\ &= 1 + \sum_{j\geq 1} \binom{n-j+1}{j} \quad . \end{aligned}$$

Proof. By Theorem 1, F_{n+2} is the number of Fibonacci subsets of $\{1, 2, \cdots, n\}$. Of these ϕ is one such subset. There are

$$\binom{n}{1}$$

singleton Fibonacci subsets of $\{1, 2, \cdots, n\}$. The two-element Fibonacci subsets are just the two-element subsets of $\{2, 3, \cdots, n\}$, and there are

$$\binom{n-1}{2}$$

of these. In general, the j-element Fibonacci subsets of $\{1, 2, \cdots, n\}$ are just the j-element subsets of $\{j, j + 1, \cdots, n\}$ and there are exactly

$$\binom{n-j+1}{j}$$

of these. Hence summing over all j and using Theorem 1, we see that

$$F_{n+2} = 1 + \sum_{j \geq 1} \binom{n-j+1}{j} .$$

Theorem 3. For $n \geq 0$

$$\binom{n+1}{1} F_1 + \binom{n+1}{2} F_2 + \cdots + \binom{n+1}{n} F_n + F_{n+1} = F_{2n+2} ,$$

or

$$\sum_{j=0}^{n} \binom{n+1}{j} F_{n+1-j} = F_{2n+2} .$$

Remark. This is the identity stated in the Introduction with $n + 1$ replacing n.

Proof. By Theorem 1, F_{2n+2} is the number of Fibonacci subsets of $\{1, 2, \cdots, 2n\}$. We first remark that there are at most n elements of a Fibonacci subset of $\{1, 3, \cdots, 2n\}$, for if there were $n + 1$ elements then at least one element would be $\leq n$ which is impossible.

Let T_j denote the number of Fibonacci subsets of $\{1, 2, \cdots, 2n\}$ that have exactly j elements $\geq n$. Clearly

$$F_{2n+2} = \sum_{j=0}^{n} T_j .$$

Now to construct the subsets enumerated by T_j, we see that we may select any j-elements in the set $\{n, n+1, \cdots, 2n\}$ and then adjoin to these j elements a j-Fibonacci subset of $\{1, 2, \cdots, n-1\}$. Since there are

$$\binom{n+1}{j}$$

choices of the j elements from $\{n, n+1, \cdots, 2n\}$ and $F_{(n-1)+2-j} = F_{n+1-j}$ j-Fibonacci subsets of $\{1, 2, \cdots, n-1\}$, we see that

$$T_j = \binom{n+1}{j} F_{n+1-j} .$$

Therefore

$$F_{2n+2} = \sum_{j=0}^{n} T_j = \sum_{j=0}^{n} \binom{n+1}{j} F_{n+1-j} .$$

Theorem 4. For $n \geq 0$,

$$1 + F_1 + F_2 + \cdots + F_n = F_{n+2} .$$

Proof. Let R_j denote the number of Fibonacci subsets of $\{1, 2, \cdots, n\}$ in which the largest element is j. Let $R_0 = 1$ in order to count the empty subset ϕ. Clearly for $j > 0$, R_j equals the number of 1-Fibonacci subsets of $\{1, 2, \cdots, j - 1\}$; thus by Theorem 1, $R_j = F_{(j-1)+2-1} = F_j$. Therefore

$$F_{n+2} = 1 + \sum_{j=1}^{n} R_j = 1 + \sum_{j=1}^{n} F_j .$$

3. CONCLUSION

The genesis of this work lies in the close relationship between the Fibonacci numbers and certain generating functions that are intimately connected with the Rogers-Ramanujan identities. Indeed if $D_{-1}(q) = D_0(q) = 1$, $D_1(q) = 1 + q$, and $D_n(q) = D_{n-1}(q) + q^n D_{n-2}(q)$, then [3; pp. 298-299]

$$D_n(q) = \sum_{j\geq 0} q^{j^2} \begin{bmatrix} n + 1 - j \\ j \end{bmatrix} , \tag{3.1}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{j=1}^{m} (1 - q^{n-j+1})(1 - q^j)^{-1}, \quad \text{for } 0 \leq m \leq n, \quad \begin{bmatrix} n \\ m \end{bmatrix} = 0 \text{ otherwise.}$$

It is not difficult to see that $D_n(q)$ is the generating function for partitions in which each part is larger than the number of parts and $\leq n$. Thus $D_n(1)$ must be F_{n+2}, the number of Fibonacci subsets of $\{1, 2, \cdots, n\}$, and this is clear from (3.1) and Theorem 2 since

$$\begin{bmatrix} n \\ m \end{bmatrix} \quad \text{equals} \quad \binom{n}{m}$$

at $q = 1$. Actually, it is also possible to prove q-analogs of Theorems 3 and 4. Namely,

$$D_{2n}(q) = \sum_{j=0}^{n+1} q^{jn} \begin{bmatrix} n + 1 \\ j \end{bmatrix} D_{n-1-j}(q) , \tag{3.2}$$

and

$$D_n(q) = 1 + \sum_{j=1}^{n} q^j D_{j-2}(q) . \tag{3.3}$$

While (3.3) is a trivial result (3.2) is somewhat tricky although a partition-theoretic analog of Theorem 3 yields the result directly.

Since $D_n(q)$ is also the generating function for partitions in which each part is $\le n$ and each part differs from every other part by at least 2, we might have defined a Fibonacci set in this way also; i.e., a finite set of positive integers in which each element differs from every other element by at least 2. Such a definition provides no new insights and only tends to make the results we have obtained more cumbersome. C. Berge [6; p. 31] gives a proof of our Theorem 2 using this particular approach.

It is to be hoped that the combinatorial approach described in this paper can be extended to prove such appealing identities as

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

[12; p. 7]

$$2^{n-1}F_n = \sum_{j\ge 0}\binom{n}{2j+1}5^j$$

[8; p. 150, e.q. (10.14.11)].

Presumably a good guide for such a study would be to first attempt (by any means) to establish the desired q-analog for $D_n(q)$. Such a result would then give increased information about the possibility of a combinatorial proof of the corresponding Fibonacci identity. This approach was used in reverse in passing from the formulae [1; p. 113]

$$F_n = \sum_{\alpha=-\infty}^{\infty}(-1)^{\alpha}\binom{n}{[1/2(n-1-5\alpha)]}$$

to new generalizations of the Rogers-Ramanujan identities ([4], [5]). I. J. Schur was the first one to extensively develop such formulas [11] (see also [2], [7]).

REFERENCES

1. George E. Andrews, "Some Formulae for the Fibonacci Sequence with Generalizations," Fibonacci Quarterly, Vol. 7, No. 2 (April 1969), pp. 113-130.
2. George E. Andrews, Advanced Problem H-138, Fibonacci Quarterly, Vol. 8, No. 1 (February 1970), p. 76.
3. George E. Andrews, "A Polynomial Identity which Implies the Rogers-Ramanujan Identities," Scripta Math., Vol. 28 (1970), pp. 297-305.

4. George E. Andrews, "Sieves for Theorems of Euler, Rogers and Ramanujan, from the Theory of Arithmetic Functions," Lecture Notes in Mathematics, No. 251, Springer, New York, 1971.
5. George E. Andrews, "Sieves in the Theory of Partitions," Amer. J. Math. (to appear).
6. C. Berge, Principles of Combinatories, Academic Press, New York, 1971.
7. Leonard Carlitz, Solution to Advanced Problem H-138, Fibonacci Quarterly, Vol. 8, No. 1 (February 1970), pp. 76-81.
8. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th Ed., Oxford University Press, Oxford, 1960.
9. V. E. Hoggatt, Jr., and Joseph Arkin, "A Bouquet of Convolutions," Proceedings of the Washington State University Conf. on Number Theory, March 1971, pp. 68-79.
10. John E. and Margaret W. Maxfield, Discovering Number Theory, W. B. Saunders, Philadelphia, 1972.
11. I. J. Schur, "Ein Beitrag zur additiven Zahlentheorie, Sitzungsber," Akad. Wissensch. Berlin, Phys.-Math. Klasse (1917), pp. 302-321.
12. N. N. Vorobyov, The Fibonacci Numbers, D. C. Heath, Boston, 1963.

FIBONACCI SUMMATIONS INVOLVING A POWER OF A RATIONAL NUMBER SUMMARY

BROTHER ALFRED BROUSSEAU
St. Mary's College, Moraga, California 94575

The formulas pertain to generalized Fibonacci numbers with given T_1 and T_2 and with

$$T_{n+1} = T_n + T_{n-1} \tag{1}$$

and with generalized Lucas numbers defined by

$$V_n = T_{n+1} + T_{n-1} \quad . \tag{2}$$

Starting with a finite difference relation such as

$$\Delta (b/a)^k T_{2k} T_{2k+2} = (b^k/a^{k+1}) T_{2k+2} (b T_{2k+4} - a T_{2k}) \tag{3}$$

values of b and a are selected which lead to a single generalized Fibonacci or Lucas number for the term in parentheses. Thus for $b = 2$, $a = 13$, the quantity in parentheses is $3T_{2k-3}$. Using the finite difference approach leads to a formula

$$\sum_{k=1}^{n} (2/13)^k T_{2k} T_{2k+5} = (1/3)\left[(2^{n+1}/13^n) T_{2n+5} T_{2n+7} - 2 T_5 T_7\right]. \tag{4}$$

Formulas are also developed with terms in the denominator.

(Continued on page 156.)