ALGORITHMS FOR THIRD-ORDER RECURSION SEQUENCES

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Given a third-order recursion relation

$$T_{n+1} = a_1 T_n - a_2 T_{n-1} + a_3 T_n \quad . \tag{1}$$

Let the auxiliary equation

$$x^3 - a_1 x^2 + a_2 x - a_3 = 0 \tag{2}$$

have three distinct roots r_1, r_2, r_3. Then any term of a sequence governed by this recursion relation can be expressed in the form

$$T_n = A_1 r_1^n + A_2 r_2^n + A_3 r_3^n \quad . \tag{3}$$

THE SEQUENCE $S_n = \sum r_i^n$

Since the individual elements of these sums are powers of the roots, the sums obey the given recursion relation. Hence it is possible to determine a few terms of S_n by means of symmetric functions and thereafter generate additional terms of the S sequence. Since this sequence is basic to all the algorithms, its generation constitutes the first algorithm. (Note. This use of the S sequence is exemplified in [1].)

ALGORITHM FOR FINDING THE TERMS OF S_n

Three consecutive terms of the sequence are:

$$\left\{ \begin{aligned} S_1 &= a_1 \\ S_2 &= a_1^2 - 2a_2 \\ S_3 &= a_1^3 - 3a_1 a_2 + 3a_3 \end{aligned} \right. \quad . \tag{4}$$

Then use the recursion relation to obtain positive and negative subscript terms of the sequence.

The algorithm will be illustrated for two recursion relations which will be used to check other algorithms numerically.

EXAMPLE 1: $x^3 - x^2 - x - 1 = 0$

n	S_n	n	S_n	n	S_n	n	S_n	n	S_n
-30	-14429	-18	47	-6	11	6	39	18	58035
-29	13223	-17	271	-5	-1	7	71	19	106743
-28	-3253	-16	-253	-4	-5	8	131	20	196331
-27	-4459	-15	65	-3	5	9	241	21	361109
-26	5511	-14	83	-2	-1	10	443	22	664183
-25	-2201	-13	-105	-1	-1	11	815	23	1221623
-24	-1149	12	43	0	3	12	1499	24	2246915
-23	2161	-11	21	1	1	13	2757	25	4132721
-22	-1189	-10	-41	2	3	14	5071	26	7601259
-21	-177	-9	23	3	7	15	9327	27	13980895
-20	795	-8	3	4	11	16	17155	28	25714875
-19	-571	-7	-15	5	21	17	31553	29	47297029
								30	86992799

EXAMPLE 2: $x^3 - 7x^2 + 5x + 4 = 0$

n	$S_n\,(-4)^n$	n	S_n
-23	2450995949 6004997927 85	0	3
-22	2879858678 8067714806 5	1	7
-21	3383761613 1827843249	2	39
		3	226
-20	3975834906 620902593	4	1359
-19	4671506147 59541201	5	8227
-18	5488902409 1011041		
-17	6449322392 180465	6	49890
-16	7577792077 14561	7	302659
		8	1836255
-15	8903714463 1313	9	11140930
-14	1046164399 5681	10	67594599
-13	1229215792 433		
-12	1444301540 49	11	410112523
-11	1697004500 9	12	2488250946
		13	1509681561 1
-10	1993985121	14	9159600445 5
-9	234271601	15	5557349493 46
-8	27532161		
-7	3232913	16	3371777360 703
-6	380577	17	2045738276 0371
		18	1241198527 21698
-5	44465	19	7530649458 07219
-4	5313	20	4569025826 000559
-3	593		
-2	81	21	2772137664 2081026
-1	5	22	1681922475 81335511
		23	1020462746 554941211
		24	6191392481 409586818
		25	3756466464 6767059627
		26	2279138391 3410171845 5
		27	1382807980 7792383837 78

RECURSION RELATIONS FOR SPACED TERMS OF A SEQUENCE

Given a sequence T_n satisfying the given recursion relation. It is desired to find the recursion relation for a spacing of k among the terms, namely, for the sequence T_{nk+a}.

Since

$$T_{nk+a} = A_1 r_1^{nk+a} + A_2 r_2^{nk+a} + A_3 r_3^{nk+a} \tag{5}$$

and since there is a change of r_i^k from one term to the next, the recursion relation is that whose roots correspond to r_i^k. Let the coefficients be given in the relation

$$x^3 - B_1 x^2 + B_2 x - B_3 = 0 .$$

Then

$$B_1 = \sum r_i^k = S_k$$
$$B_2 = \sum r_i^k r_j^k = a_3^k \sum r_i^{-k} = a_3^k S_{-k}$$
$$B_3 = a_3^k ,$$

Hence the recursion relation is given by

$$x^3 - S_k x^2 + a_3^k S_{-k} - a_3^k = 0 . \tag{6}$$

EXAMPLE FOR $x^3 - x^2 - x - 1 = 0$ with $k = 5$.

$$T_{n+5} = 21 T_n + T_{n-5} + T_{n-10} .$$

Using the sequence S_n with $n = 20$,

$$T_{25} = 21*196331 + 9327 + 443 = 4132721 .$$

EXAMPLE FOR $x^3 - 7x^2 + 5x + 4 = 0$ using the terms of the S sequence.

$$T_{-5} = (-593 T_{-2} + 226 T_1 - T_4)/64$$
$$T_{-5} = (-593*81/16 + 226*7 - 1359)/64 = -44465/1024 .$$

SECOND-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

If there are several sequences satisfying the given recursion relation, a sum of terms of the form $T_{m_1}^{(1)} T_{m_2}^{(2)}$ would form a homogeneous sequence function of the second degree. Such terms if expanded using the roots of the auxiliary equation would yield terms of the form $B_i r_i^{m_1+m_2}$ and others of the form $C_{ij} r_i^{m_1} r_j^{m_2}$. The first type obey the recursion relation for r_i^2 since there is a change of 2 in the power in going from one term in the product to the next as the m's change by 1. The second type obey the recursion relation for the quantities $r_i r_j$.

ALGORITHM FOR THE SECOND-DEGREE FUNCTIONS

The recursion relation governing the quantities r_i^2 has already been obtained and is given by:

$$x^3 - S_2 x^2 + a_3^2 S_{-2} x - a_3^2 = 0 . \tag{7}$$

For the second we need to find the symmetric functions of the roots $r_i r_j$.

$$\begin{aligned} B_1 &= \sum r_i r_j = a_2 \\ B_2 &= \sum r_i^2 r_j r_k = a_3 a_1 \\ B_3 &= r_i^2 r_j^2 r_k^2 = a_3^2 \quad . \end{aligned}$$

Hence the recursion relation is

$$x^3 - a_2 x^2 + a_3 a_1 x - a_3^2 = 0 \; . \tag{8}$$

The total recursion relation is the product of (7) and (8):

$$(x^3 - S_2 x^2 + a_3^2 S_{-2} x - a_3^2)(x^3 - a_2 x^2 + a_3 a_1 x - a_3^2) = 0 \; . \tag{9}$$

EXAMPLE FOR $x^3 - 7x^2 + 5x + 4 = 0$.

$$\begin{aligned} S_5^2 &= 44 S_4^2 - 248 S_3^2 - 655 S_2^2 + 1564 S_1^2 + 848 S_0^2 - 256 S_{-1}^2 \\ &= 44*1359^2 - 248*226^2 - 655*39^2 + 1564*7^2 + 848*3^2 + 256*(5/4)^2 \\ &= 67683529 = 8227^2 \quad . \end{aligned}$$

THIRD-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

An expression of the form

$$T_{m_1}^{(1)} T_{m_2}^{(2)} T_{m_3}^{(3)}$$

gives rise to terms of the form

$$r_i^{m_1+m_2}, \qquad r_i^{m_1+m_2} r_j^{m_3}, \qquad r_i^{m_1} r_j^{m_2} r_k^{m_3} \quad .$$

The first type corresponds to the recursion relation for r_i^3, the second to the recursion relation for $r_i^2 r_j$, and the third to the recursion relation for a_3. The first relation is:

$$x^3 - S_3 x^2 + a_3^3 S_{-3} x - a_3^3 = 0 \; . \tag{10}$$

The last relation is:

$$x - a_3 = 0 \; . \tag{11}$$

For the second we have a relation of the sixth degree with coefficients symmetric functions of the roots

$R_1 = r_1^2 r_2$, $\quad R_2 = r_2^2 r_1$, $\quad R_3 = r_1^2 r_3$, $\quad R_4 = r_3^2 r_1$, $\quad R_5 = r_2^2 r_3$, $\quad R_6 = r_3^2 r_2$.

$$B_1 = \sum R_i = (21) = -3a_3 + a_2 a_1 ,$$

where the notation $(21) = \sum r_i^2 r_j$ taken as a symmetric function.

(12)
$$B_2 = \sum R_i R_j = (41^2) + (3^2) + (321) + 3(222)$$
$$B_2 = 6a_3^2 - 5a_3 a_2 a_1 + a_3 a_1^3 + a_2^3$$
$$B_3 = \sum R_i R_j R_k = (531) + 2(432) + 2(3^3)$$
$$B_3 = -7a_3^3 + 6a_3^2 a_2 a_1 - 2a_3^2 a_1^3 - 2a_3 a_2^3 + a_3 a_2^2 a_1^2$$
$$B_4 = (63^2) + (5^2 2) + (543) + 3(444) = a_3^3(3) + a_3^2(3^2) + a_3^3(21) + 3(4^3)$$
$$B_4 = 6a_3^4 - 5a_3^3 a_2 a_1 + a_3^3 a_1^3 + a_3^2 a_2^3$$
$$B_5 = (654) = a_3^4(21) = -3a_3^5 + a_3^4 a_2 a_1$$
$$B_6 = a_3^6$$

The product of (10), (11) and the polynomial whose coefficients are given by (12) is the required recursion relation for the third degree. APPLIED TO $x^3 - x^2 - x - 1 = 0$, we have

$$(x^3 - 7x^2 + 5x - 1)(x - 1)(x^6 + 4x^5 + 11x^4 + 12x^3 + 11x^2 + 4x + 1) = 0$$

or

$$x^{10} - 4x^9 - 9x^8 - 34x^7 + 24x^6 - 2x^5 + 40x^4 - 14x^3 - x^2 - 2x + 1 = 0 .$$

Starting with $S_9 = 241$ we have:

$$4*241^3 + 9*131^3 + 34*71^3 - 24*39^3 + 2*21^3 - 40*11^3 + 14*7^3 + 3^3 + 2*1^3 - 3^3$$
$$= 86938307 = 443^3 \cdot S_{10}^3 .$$

FOURTH-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

We proceed as before but without going through the preliminary details we arrive at the conclusion that the symmetric functions of the roots are given by the partitions (4), (31), (22), (211) of four into three parts or less. We determine the recursion relations or equivalently the coefficients for each of these.

<u>(4)</u>

(13) $$x^3 - S_4 x^2 + a_3^4 S_{-4} x - a_3^4 = 0 .$$

<u>(211)</u> Since this symmetric function is equivalent to $a_3 r_i$ in its terms, we have the relation

(14) $$x^3 - a_3 a_1 x^2 + a_3^2 a_2 x - a_3^4 = 0 .$$

<u>(31)</u>

$$A_1 = (31) = -a_3 a_1 - 2a_2^2 + a_2 a_1^2$$

$$A_2 = (611) + (44) + (431) + (332)$$

$$= a_3(5) + (44) + a_3(32) + a_3^2 a_2$$

$$A_2 = -a_3^2 a_2 + 5a_3^2 a_1^2 + 2a_3 a_2^2 a_1 - 5a_3 a_2 a_1^3 + a_3 a_1^5 + a_2^4$$

$$A_3 = (741) + (642) + (543) + 2(444)$$

$$= a_3(63) + a_3^2(42) + a_3^3(21) + 2a_3^4$$

(15)

$$A_3 = 2a_3^4 - 13a_3^3 a_2 a_1 + a_3^3 a_1^3 + a_3^2 a_2^3 + 10a_3^2 a_2^2 a_1^2 - 3a_3^2 a_2 a_1^4 - 3a_3 a_2^4 a_1 + a_3 a_2^3 a_1^3$$

$$A_4 = a_3^2(5^2) + a_3^4(4) + a_3^4(31) + a_3^5(1)$$

$$A_4 = -a_3^5 a_1 + 5a_3^4 a_2^2 + 2a_3^4 a_2 a_1^2 - 5a_3^3 a_2^3 a_1 + a_3^2 a_2^5 + a_3^4 a_1^4$$

$$A_5 = a_3^5(32) = -a_3^6 a_2 - 2a_3^6 a_1^2 + a_3^5 a_2^2 a_1$$

$$A_6 = a_3^8$$

<u>(22)</u>

$$B_1 = (2^2) = -2a_3 a_1 + a_2^2$$

(16)

$$B_2 = (422) = a_3^2(2) = -2a_3^2 a_2 + a_3^2 a_1^2$$

$$B_3 = a_3^4 \quad .$$

The product of the polynomials given by (13), (14), (15), and (16) gives the required recursion relation for the fourth degree.

APPLICATION TO $x^3 - x^2 - x - 1 = 0$.

$$(x^3 - 11x^2 - 5x - 1)(x^6 + 4x^5 + 15x^4 - 24x^3 + 7x^2 + 1)(x^3 + x^2 + 3x - 1) \times (x^3 - x^2 - x - 1) = 0$$

or

$$x^{15} - 7x^{14} - 33x^{13} - 223x^{12} + 197x^{11} + 41x^{10} + 1559x^9 - 451x^8 - 373x^7 - 637x^6 + 269x^5 + 131x^4 + 47x^3 - 5x^2 - 3x - 1 = 0 \ .$$

REMARKS

The determination of the coefficients of the polynomials for higher degrees in terms of the coefficients of the original recursion relation leads to expressions of ever greater complexity which make calculations tedious and present a greater possibility of error. A simpler approach is to use symmetric functions of the roots which in turn can be calculated by means of the S sequence of the given recursion relation. For three roots all such symmetric functions can be reduced to one of the forms (ab), (a^2) or (a). The last is simply S_a while the others are given by:

(17) $$(ab) = S_a S_b - S_{a+b}$$

(18) $$(a^2) = (S_a^2 - S_{2a})/2 .$$

FIFTH-DEGREE HOMOGENEOUS SEQUENCE FUNCTIONS

On the basis of partitions we consider symmetric functions of the roots of the forms (5), (41), (32), (311), (221).

<u>(5)</u>

$$x^3 - S_5 x^2 + a_3^5 S_{-5} x - a_3^5 .$$

<u>(311)</u>

$$B_1 = a_3(2)$$
$$B_2 = a_3^2(2^2)$$
$$B_3 = a_3^5 .$$

<u>(221)</u>

$$C_1 = a_3 a_2$$
$$C_2 = a_3^3 a_1$$
$$C_3 = a_3^5 .$$

<u>(41)</u>

$$D_1 = (41)$$
$$D_2 = (81^2) + (5^2) + (541) + (442)$$
$$= a_3(7) + (5^2) + a_3(43) + a_3^2(2^2)$$
$$D_3 = (951) + (852) + (654) + 2(5^3)$$
$$= a_3(84) + a_3^2(63) + a_3^4(21) + 2a_3^5$$
$$D_4 = (992) + (10,55) + (965) + (866)$$
$$= a_3^2(7^2) + a_3^5(5) + a_3^5(41) + a_3^6(2)$$
$$D_5 = (10,96) = a_3^6(43)$$
$$D_6 = a_3^{10} .$$

<u>(32)</u>

$$E_1 = (32)$$
$$E_2 = (622) + (55) + (532) + (433)$$
$$= a_3^2(4) + (5^2) + a_3^2(31) + a_3^3 a_1$$
$$E_3 = (852) + (654) + 2(555) + (753)$$
$$= a_3^2(63) + a_3^4(21) + 2a_3^5 + a_3^3(42)$$
$$E_4 = (884) + (10,55) + (875) + (776)$$
$$= a_3^4(4^2) + a_3^5(5) + a_3^5(32) + a_3^6 a_2$$
$$E_5 = (10,87) = a_3^7(31)$$
$$E_6 = a_3^{10} .$$

APPLICATION TO $x^3 - x^2 - x - 1 = 0$.

$$(x^3 - 21x^2 - x - 1)(x^3 - 3x^2 - x - 1)(x^3 + x^2 + x - 1)(x^6 + 0x^5 + 7x^4 - 24x^3 + 15x^2 + 4x + 1)$$
$$\times (x^6 + 10x^5 + 75x^4 + 28x^3 - x^2 - 6x + 1) = 0 .$$

The product is

$$x^{21} - 13x^{20} - 110x^{19} - 1374x^{18} \pm 2425x^{17} + 543x^{16} + 60340x^{15} - 3976x^{14}$$
$$- 43106x^{13} - 149310x^{12} + 137592x^{11} + 88200x^{10} + 63126x^9 - 21742x^8 - 13076x^7$$
$$- 8932x^6 + 1041x^5 - 37x^4 + 150x^3 - 10x^2 + x - 1 = 0 .$$

CONCLUDING NOTES

1. That the symmetric functions of the roots can always be expressed in terms of the quantities S_n is an elementary proposition in combinatorial analysis. (See [2; p. 7].)

2. For the n^{th} degree, the recursion relation has degree

$$\binom{n+2}{2} .$$

This follows from the fact that the number of terms involving the roots is equivalent to the solution of $x + y + z = n$ in positive integers and zero.

3. For the sixth-degree relations, the coefficients A_1 and A_5, A_2 and A_4, are complementary, the respective quantities in the symmetric functions adding up to $2n$.

4. Each term in a coefficient has a weight. The coefficient A_k would have its terms of weight kn where n is the degree being considered for the terms of the original recursion relation. Thus for $n = 8$, E_4 has a term $a_3^6(7^2)$ which has a weight $6 \times 3 + 2 \times 7 = 32 = 4 \times 8$.

5. If $a_3 = 1$, all the factors for the n^{th} degree are found for degree $n + 3$.

6. With some modifications on the symmetric functions involved, this approach could be used to produce algorithms relating to recursion relations of higher order.

7. The algorithms were checked numerically by using a relation with roots 1, 2, and 4, finding the symmetric functions directly and comparing the result with that given by the algorithms.

REFERENCES

1. Trudy Y. H. Tong, "Some Properties of the Tribonacci Sequence and the Special Lucas Sequence," Master's Thesis, San Jose State University, August 1970.
2. P. A. MacMahon, Combinatory Analysis, Cambridge University Press, 1915, 1916. Reprinted by Chelsea Publishing Company, 1960.

Editor's Note: There are an additional twelve pages on this subject, going through the tenth degree. If you would like a Xerox copy of the additional material at four cents a page (which includes postage, materials and labor), send your request to:

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