ON FERNS' THEOREM ON THE EXPANSION OF FIBONACCI AND LUCAS NUMBERS

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Let (F_n) be a Fibonacci-type integer sequence satisfying the recurrence relation $F_n = pF_{n-1} + qF_{n-2}$ $(n \geqslant 2)$ in which $p^2 + 4q \neq 0$, and let (L_n) be the corresponding Lucas-type sequence, as described in [2]. The object of this note is both to generalize Ferns' theorem [1] on the expansion of

$$F_{x_1+x_2+\cdots+x_n} \qquad \text{and} \qquad L_{x_1+x_2+\cdots+x_n}$$

and to simplify the proof. Ferns' theorem was proved for the case when (F_n) and (L_n) were the Fibonacci and Lucas sequences, respectively, so in the statement and proof of the theorem the reader may interpret (F_n) and (L_n) as the ordinary Fibonacci and Lucas sequences, if he so desires.

Let

$$S_k^n = \Sigma F_{x_{i_1}} F_{x_{i_2}} \cdots F_{x_{i_k}} L_{x_{j_1}} \cdots L_{x_{j_{n-k}}}$$

where the sum ranges over all permutations $(i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$ of $(1, \cdots, n)$ such that

$$1 \leqslant i_1 < i_2 \cdots < i_k \leqslant n \qquad \text{and} \qquad 1 \leqslant j_1 < j_2 < \cdots < j_{n-k} \leqslant n,$$

for $0 \leqslant k \leqslant n$. Let α and β be the roots of $x^2 - px - q$ and let $A = F_1 - F_0\beta$, $B = F_1 - F_0\alpha$. Then $A \neq 0$ and $B \neq 0$ (see [2]) so that

$$\alpha = \left(\frac{L_1 + dF_1}{2A}\right), \qquad \beta = \left(\frac{L_1 - dF_1}{2B}\right),$$

where

$$d = \sqrt{p^2 + 4q}\ .$$

Then the generalized version of Ferns' theorem may be stated in the following way.

Theorem: If

$$\Sigma_e = S_0^n + d^2S_2^n + d^4S_4^n + \cdots \qquad \text{and} \qquad \Sigma_o = dS_1^n + d^3S_3^n + d^5S_5^n + \cdots$$

then

$$F_{x_1+x_2+\cdots+x_n} = \frac{1}{2^n d}\left\{\left(\frac{1}{A^{n-1}} - \frac{1}{B^{n-1}}\right)\Sigma_e + \left(\frac{1}{A^{n-1}} + \frac{1}{B^{n-1}}\right)\Sigma_o\right\}$$

and

$$L_{x_1+x_2+\cdots+x_n} = \frac{1}{2^n}\left\{\left(\frac{1}{A^{n-1}} + \frac{1}{B^{n-1}}\right)\Sigma_e + \left(\frac{1}{A^{n-1}} - \frac{1}{B^{n-1}}\right)\Sigma_o\right\}.$$

Proof: It is well known that if r is a positive integer

$$F_r = \frac{A\alpha^r - B\beta^r}{\alpha - \beta}, \qquad L_r = A\alpha^r + B\beta^r\ .$$

Therefore,

$$\alpha^r = \frac{L_r + dF_r}{2A}, \qquad \beta^r = \frac{L_r - dF_r}{2B}$$

Therefore

$$\frac{1}{2A}(L_{x_1+x_2+\cdots+x_n} + dF_{x_1+x_2+\cdots+x_n})$$

$$= \alpha^{x_1+x_2+\cdots+x_n}$$

$$= \frac{1}{2^n A^n}(L_{x_1} + dF_{x_1})(L_{x_2} + dF_{x_2}) \cdots (L_{x_n} + dF_{x_n})$$

$$= \frac{1}{2^n A^n}(S_0^n + dS_1^n + d^2 S_2^n + \cdots + d^n S_n^n).$$

Similarly

$$\frac{1}{2B}(L_{x_1+x_2+\cdots+x_n} - dF_{x_1+x_2+\cdots+x_n})$$

$$= \frac{1}{2^n B^n}(S_0^n - dS_1^n + d^2 S_2^n - \cdots + (-1)^n d^n S_n^n).$$

The theorem now follows by addition and subtraction.

REFERENCES

1. H.H. Ferns, "Products of Fibonacci and Lucas Numbers," *The Fibonacci Quarterly,* Vol. 7, No. 1 (Feb. 1969), pp. 1–13.
2. A.J.W. Hilton, "On the Partition of Horadam's Generalized Sequences into Generalized Fibonacci and Lucas Sequences," *The Fibonacci Quarterly,* to appear.

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THE FIBONACCI ASSOCIATION

RESEARCH CONFERENCE

PROGRAM OF SATURDAY, MAY 4, 1974

ST. MARY'S COLLEGE

9:00–9:30	PRELIMINARY GATHERING, coffee and rolls.
9:30–10:15	SEQUENCES GENERATED BY LEAST INTEGER FUNCTIONS Brother Alfred Brousseau, St. Mary's College
10:20–11:00	THE SEQUENCES 1, 5, 16, 45, 121, 320, ··· IN COMBINATORICS Ken Rebman, California State University, Hayward
11:05–11:45	REPRESENTATION OF INTEGERS USING FIBONACCI AND LUCAS SQUARES Hardy Reyerson, Masters Student, San Jose State University
12:00–1:30	LUNCH PERIOD
1:30–2:15	RECTANGULAR AND TRIANGULAR PARTITIONS Leonard Carlitz, Duke University
2:20–3:00	GREAT ADVENTURES WITH CATALAN AND LAGRANGE Verner E. Hoggatt, Jr., San Jose State University

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