$$5\sum_{k=0}^{n}(-1)^k kH_{2k+1} = (-1)^n(nH_{2n+3}+(n+1)H_{2n+1})-p \tag{4.14}$$

$$4\sum_{k=0}^{n}(-1)^k kH_{m+3k} = 2(-1)^n(n+1)H_{m+3n+1}-(-1)^nH_{m+3n+2}-H_{m-1} \quad (m = 2, 3, \cdots) \tag{4.15}$$

and so on.

REFERENCES

1. L. Carlitz and H. H. Ferns, "Some Fibonacci and Lucas Identities," *The Fibonacci Quarterly,* Vol. 8, No. 1 (Feb. 1970), pp. 61-73.
2. L. E. Dickson, *History of the Theory of Numbers,* Vol. 1, New York, 1952, pp. 393-407.
3. V. C. Harris, "Identities Involving Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 3, No. 3 (Oct. 1965), pp. 214–218.
4. M.R. Iyer, "Identities Involving Generalized Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 7, No. 1 (Feb. 1969), pp. 66–73.
5. V. E. Hoggatt, Jr., and M. Bicknell, "Fourth-Power Fibonacci Identities from Pascal's Triangle," *The Fibonacci Quarterly,* Vol. 2, No. 4 (Dec. 1964), pp. 261–266.
6. A.F. Horadam, "A Generalized Fibonacci Sequence, *Amer. Math. Monthly,* Vol. 68, No. 5, 1961, pp. 455-459.
7. A. F. Horadam, "Basic Properties of a Certain Generalized Sequence of Numbers," *The Fibonacci Quarterly,* Vol. 3, No. 3 (Oct. 1965), pp. 161-176.
8. K. Subba Rao, "Some Properties of Fibonacci Numbers," *Amer. Math. Monthly,* Vol. 60, No. 10, 1953, pp. 680-684.
9. J. E. Walton and A. F. Horadam, "Some Aspects of Generalized Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct. 1974), pp. 241–250.
10. D. Zeitlin, "On Identities for Fibonacci Numbers," *Amer. Math. Monthly,* Vol. 70, No. 11, 1963, pp. 987–991.
11. D. Zeitlin, "Power Identities for Sequences Defined by $W_{n+2} = dW_{n+1} - cW_n$," *The Fibonacci Quarterly,* Vol. 3, No. 4 (Dec. 1965), pp. 241–256.

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[Continued from Page 271.]

where X is the largest root of

$$x^4 - x^3 - 3x^2 + x + 1 = 0. \tag{3}$$

The astonishing appearance of (1) stems from a peculiarity of (3). The Galois group of this quartic is the octic group (the symmetries of a square), and its resolvent cubic is therefore reducible:

$$z^3 - 8z - 7 = (z+1)(z^2 - z - 7) = 0. \tag{4}$$

The common discriminant of (3) and (4) equals $725 = 5^2 \cdot 29$. While the quartic field $Q(X)$ contains $Q(\sqrt{5})$ as a subfield it does not contain $Q(\sqrt{29})$. Yet X can be computed from any root of (4). The rational root $z = -1$ gives $X = (A+1)/4$ while $z = (1+\sqrt{29})/2$ gives $X = (B+1)/4$.

It is clear that we can construct any number of such incredible identities from other quartics having an octic group. For example

$$x^4 - x^3 - 5x^2 - x + 1 = 0$$

has the discriminant $4205 = 29^2 \cdot 5$, and so the two expressions involve $\sqrt{5}$ and $\sqrt{29}$ once again. But this time $Q(\sqrt{29})$ is in $Q(X)$ and $Q(\sqrt{5})$ is not.

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