ARGAND DIAGRAMS OF EXTENDED FIBONACCI AND LUCAS NUMBERS

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Numerous extensions of the Fibonacci and Lucas Numbers have been reported in the literature [1–6]. In this paper we present a computer-generated plot of the complex representation of the Fibonacci and Lucas Numbers. The complex representation of the Fibonacci Numbers is given by [5,6].

$$F(x) = \frac{\phi^x - \phi^{-x}[\cos(x\pi) + i\sin(x\pi)]}{\sqrt{5}},$$

where

$$\phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad F(-x) = (-1)^{n+1}F(x),$$

$$Re[F(x)] = \frac{1}{\sqrt{5}}\left\{\phi^x - \phi^{-x}\cos(\pi x)\right\};$$

and

$$Im[F(x)] = \frac{1}{\sqrt{5}}\left\{-\phi^{-x}\sin(\pi x)\right\}$$

The Fibonacci identity: $F(x) = F(x-1) + F(x-2)$ is preserved for the complex parts of $F(x)$:

$$Re[F(x)] = Re[F(x-1)] + Re[F(x-2)]$$

and

$$Im[F(x)] = Im[F(x-1)] + Im[F(x-2)].$$

Figure 1 is a computer-generated Argand plot of $F(x)$ in the range $-5 < x < +5$.

The branch of the curve for positive x approaches the real axis as x increases. Defining the tangent angle of the curve as:

$$\psi = \tan^{-1}\left\{\frac{Im[F(x)]}{Re[F(x)]}\right\};$$

this angle approaches zero for large positive x since

$$\lim_{x\to\infty} = Im[F(x)] = 0.$$

The negative branch of the curve approaches a logarithmic spiral for x large and negative. The modulus r is given by:

$$r = \left\{Re^2[F(x)] + Im^2[F(x)]\right\}^{1/2}$$

in the limit

$$r \approx \frac{\phi^{-x}}{\sqrt{5}}; \qquad \psi \approx \pi x, \qquad r \approx \frac{1}{\sqrt{5}}\left\{\phi^{-\psi/\pi}\right\};$$

therefore,

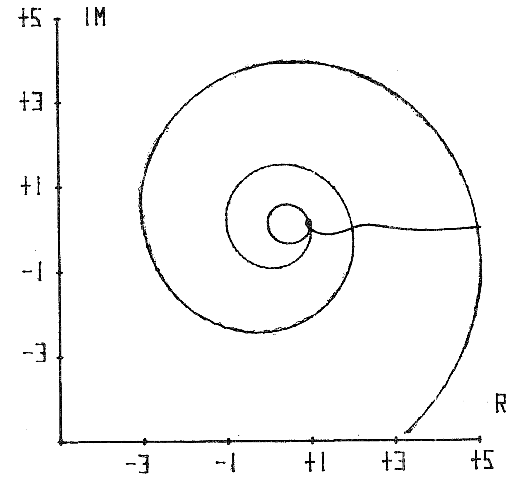

Fig. 1 Computer-Generated Argand Plot of the Fibonacci Function

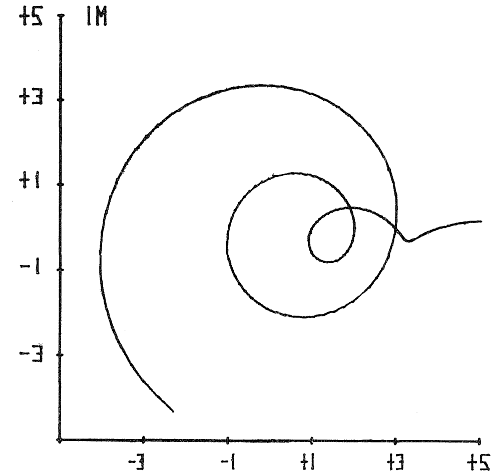

Fig. 2 Computer-Generated Argand Plot of the Lucas Function

$$\ln r \approx (-\psi/\pi)k\ ,$$

where

$$k = \ln(\phi/\sqrt{5}) \qquad \text{and} \qquad r \approx e^{-(\psi k/\pi)} = e^{-kx}\ .$$

Similarly, the Lucas number identity:

$$L(x) = F(x+1) + F(x-1)$$

leads directly to [6]:

$$L(x) = \phi^x + (-1)^x \phi^{-x}$$

and the complex representation of the Lucas Numbers follows

$$L(x) = \phi^x + \phi^{-x}(\cos \pi x + i \sin \pi x)$$

with

$$Re[L(x)] = \phi^x + \phi^{-x} \cos \pi x \qquad \text{and} \qquad Im[L(x)] = \phi^{-x} \sin \pi x\ .$$

Note:

$$Im[L(x)] = \frac{-1}{\sqrt{5}}\, Im[F(x)]\ .$$

As with the previous case for n large and positive, the positive branch of the Lucas number curve approaches the Real axis. Again, the negative branch approaches a logarithmic spiral for n large and negative.

$$\psi \approx \pi x, \quad r \approx \phi^{-(\psi/\pi)}, \quad \ln r \approx -(\psi/\pi)\ln \phi, \quad r \cong e^{-(\psi/\pi)\phi} = e^{-\phi x}.$$

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