# FIBONACCI NUMBERS IN TREE COUNTS FOR SECTOR AND RELATED GRAPHS

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## 1. INTRODUCTION

Consider a set of $(n+1)$ vertices of a nonoriented graph with vertices $1, 2, \cdots, n$ adjacent to vertex $(n+1)$ and with vertex $i$ adjacent to vertex $(i+1)$ for $1 \leqslant i \leqslant n-1$. The graph described is called a *sector* graph herein. If the first $n$ vertices are equi spaced in clockwise ascending order on the circumference of a circle with the $(n+1)^{st}$ vertex at the center, the geometry justifies the choice of name.

If the first and $n^{th}$ vertices were made adjacent, the result would be the well known *wheel**, $W_{n+1}$ described by Harary [1]. For this reason, the lucidly descriptive terminology of $W_{n+1}$ is applicable to a sector graph as well. Vertex $(n+1)$ is a *hub* vertex with *spokes* radiating outward to the $n$ *rim vertices* which are adjacent by virtue of *rim* edges. Multiple spokes and/or rim edges are admissible.

The designation used herein for a sector graph with $n$ rim vertices is $S_n$ followed in parentheses by spoke and rim edge multiplicity information. (In particular, rim edge position $i$ is between vertices $i$ and $(i+1)$. This would also specify the position of sector $i$.) Thus, the designation $S_8(1(2), 6(3), \boxed{3}(2))$ would describe a sector graph of nine vertices total having double spokes in the first spoke position, triple spokes in the sixth spoke position, and double rim edges in the third rim edge position. A simple sector graph would only require the designation $S_n$. The same terminology applies to wheels after a rim vertex is designated as vertex 1. An example is given in Figure 1. The

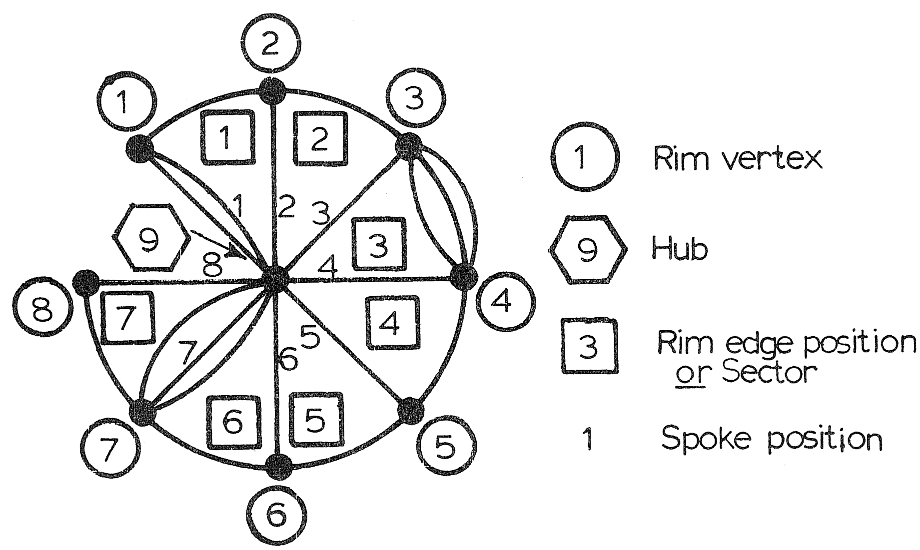


Figure 1 Example of $S_8(1(2), 6(3), \boxed{3}(3))$

number of trees is indicated by prefixing a $T$. Thus, $TS_n$ is the number of trees in a simple sector graph. (Unless otherwise stated, trees will refer to *spanning* trees.)

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*The subscript for the wheel customarily denotes the total number of vertices including the hub. The subscript $n+1$ is used here to retain identification with the $n$ rim vertices.

## 2. THE COUNT OF $TS_n(1(2), n(2))$ AND A BASIC DETERMINANT

If a graph has some measure of symmetry, an algebraic approach to counting of trees is often feasible. If one row of the incidence matrix $A$ of the graph is suppressed to obtain the reduced incidence matrix $A_n$ (of rank $n$), it is known [2] that the number of trees is given by $det(A_nA_n^t)$, where $t$ indicates the transpose operation. In the case of $S_n(1(2), n(2))$, suppressing the hub vertex row yields

$$(1)\qquad det(A_nA_n^t) = TS_n(1(2), n(2)) = \underbrace{\begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}}_{n \times n} = a_n\,.$$

The determinant $a_n$ of (1) is basic to succeeding work.

The recurrence realtion from (1) is easily found to be

$$(2)\qquad a_n = 3a_{n-1} - a_{n-2}$$

whose solution is [3]

$$(3)\qquad a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right].$$

Physically, (3) is valid for $n \geqslant 2$. However, $a_0 = 1$, $a_1 = 3$ are consistent mathematically. The resulting numerical sequence of tree counts is

$$(4)\qquad 1, 3, 8, 21, 55, 144, \cdots \qquad (n = 0, 1, 2, 3, 4, 5, \cdots).$$

It is evident that (4) gives alternate numbers of the Fibonacci sequence

$$(5)\qquad F_1, F_2, F_3, F_4, F_5, F_6, \cdots \to 1, 1, 2, 3, 5, 8, \cdots.$$

Upon comparing (5) with (4), it is seen that

$$(6)\qquad a_n = TS_n(1(2), n(2)) = F_{2n+2}\,.$$

This result is not surprising, of course, since it is well known [4] that electrical ladder networks have graphs of the sector type and immittance calculations on unit element ladders involve tree-derived numerators and denominators of Fibonacci numbers.

Application of the $Z$-Transform [5] to (2), results in

$$(7)\qquad Z(a_n) = \frac{z^2a_0 + z(a_1 - 3a_0)}{z^2 - 3z + 1}\,.$$

By dividing the numerator of (7) by the denominator, the values of $a_n$ are found as coefficients of $1/z^n$. By setting $a_0 = 1$, $a_1 = 3$,

$$(8)\qquad Z(a_n) = \frac{z^2}{z^2 - 3z + 1}$$

is found as the generating function in powers of $1/z$ of the sequence (4).

## 3. THE COUNT OF $TS_n(1(2))$

Next, consider $TS_n(1(2))$ (by symmetry, $TS_n(n(2))$. $Det(A_nA_n^t)$ is the same as that of (1) except the first 3 on the main diagonal is replaced by 2. Thus, in terms of $a_n$ and through the use of (2) and (6),

$$(9)\qquad TS_n(1(2)) = 2a_{n-1} - a_{n-2} = a_n - a_{n-1} = F_{2n+2} - F_{2n} = F_{2n+1}\,.$$

The Fibonacci numbers not in (4) satisfy the same recurrence relation (2) as those in (4). Use of new initial conditions with (2), say, $a_n = 5$ for $n = 2$ and $a_n = 13$ for $n = 3$ yield

$$TS_n(1(2)) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n = F_{2n+1}\,. \tag{10}$$

The resulting sequence of tree count numbers is

$$1, 2, 5, 13, 34, 89, \cdots \qquad (n = 0, 1, 2, 3, 4, 5, \cdots), \tag{11}$$

where physical validity applies for $n \geqslant 2$.

By letting $a_0 = 1$, $a_1 = 2$ in (7), the generating function for the sequence (11) becomes

$$Z(TS_n(1(2))) = \frac{z^2 - z}{z^2 - 3z + 1}\,. \tag{12}$$

## 4. THE COUNT OF $TS_n$

In $S_n$, the degree of rim vertices $1$ and $n$ is two. Hence, the $\det(A_nA_n^t)$ for $S_n$ is the same as (1) except that the 3's in the $(1,1)$ and $(n,n)$ positions are replaced by 2's. There results

$$TS_n = \underbrace{\begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & & & & & & 0 \\ 0 & & & & & & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & a_{n-2} & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & & & & & & 0 \\ 0 & & & & & & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{vmatrix}}_{n \times n} = 2\begin{vmatrix} & & & 0 \\ & & & 0 \\ & & & 0 \\ & a_{n-2} & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & -1 \\ 0\ 0\ 0 & \cdots & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 & \cdots & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & \cdot & \cdot \\ & & a_{n-3} & & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & 1 & 0 \\ & & & & 3 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{vmatrix} \tag{13}$$

$$= 4a_{n-2} - 4a_{n-3} + a_{n-4} = a_{n-1} = F_{2n}\,.$$

This means that

$$TS_n = TS_{n-1}(1(2), n-1(2)) = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]\,. \tag{14}$$

An index shift by one can be accomplished in (8) by multiplication by $1/z$. Hence, the generating function for $TS_n$ becomes

$$Z(TS_n) = \frac{z}{z^2 - 3z + 1}\,. \tag{15}$$

In terms of *sectors*, the simple sector graph of $k$ sectors has the tree count given by (6) with $n$ replaced by $k$.

## 5. EXTENSION TO $TW_{n+1}$

In $S_n$, by additionally making rim vertices $1$ and $n$ simply adjacent, the simple wheel $W_{n+1}$ is obtained. $Det(A_nA_n^t)$ is the same as (1) except that $-1$'s replace $0$'s in the $(1,n)$ and $(n,1)$ positions. There results

$$TW_{n+1} = \begin{vmatrix} & & & & & -1 \\ & & & & & 0 \\ & & & & & 0 \\ & & a_{n-1} & & & \vdots \\ & & & & & 0 \\ & & & & & -1 \\ -1 & 0 & 0 & \cdots\ 0 & -1 & 3 \end{vmatrix} = 3a_{n-1} - 2a_{n-2} - 2 = a_n - a_{n-2} - 2 \tag{16}$$

$$= \frac{3}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n - \frac{2\sqrt{5}}{3}\right] = F_{2n+2} - F_{2n-2} - 2.$$

## 6. COUNT OF TREES WHICH INCLUDE INDIVIDUAL SPOKES OR RIM EDGES

One way to find the number of trees which contain a particular graph edge is to coalesce the vertices of the edge and count the trees of the vertex-reduced graph, the count being the desired number of trees in the unreduced graph [6] containing the edge. The self loop into which the edge degenerates can be disregarded for tree counting.

If a connected graph is separable, the number of trees is equal to the product of the trees of the separable subgraphs. When removal of a graph edge produces two separable but connected components, the difference between the product tree count and the number of trees of the original graph provides an additional way of finding the number of trees containing a particular graph edge. A few easily extended illustrative examples follow.

COUNT OF TREES WITH A GIVEN SPOKE. Consider the $h^{th}$ spoke of $S_n$. By coalescing vertex $h$ with hub vertex, two edge-disjoint subgraphs appear so that the vertex-reduced graph is separable with the hub vertex being a cut vertex. Each subgraph is a sector graph having a double end spoke. One subgraph has $(h-1)$ vertices and the other has $(n-h)$ vertices. Through use of (9) and the product rule for separable graphs, it is seen that the number of trees of $S_n$ which contain spoke $h$ is

$$(17) \qquad T_{h-1}(1(2)) \cdot T_{n-h}(1(2)) = F_{2h-1} \cdot F_{2n-2h+1}, \qquad (1 \leqslant h \leqslant n) .$$

Consider *any* spoke of $W_n$. Coalescing the rim vertex to the hub yields $S_{n-1}(1(2), n-1(2))$ which, by (6), has

$$(18) \qquad TS_{n-1}(1(2), n-1(2)) = F_{2n}$$

trees. Thus, any spoke of $W_n$ is in $F_{2n}$ trees.

COUNT OF TREES WITH A GIVEN RIM EDGE. Let rim edge $k$ be the edge of $S_n$ which is incident with rim vertices $k$ and $(k+1)$. Removal of rim edge $k$ reduces $S_n$ to a separable graph having the hub vertex as the cut vertex. The subgraphs are the sector graphs $S_k$ and $S_{n-k}$. They are

$$(19) \qquad TS_k \cdot TS_{n-k} = F_{2k} \cdot F_{2n-2k}$$

trees in the reduced graph. Since $S_n$ has $F_{2n}$ trees, the number of trees of $S_n$ in which rim edge $k$ appears is

$$(20) \qquad TS_n - TS_k \cdot TS_{n-k} = F_{2n} - F_{2k} \cdot F_{2n-2k} .$$

If any rim section is removed from $W_n$, $S_n$ results. Therefore, any rim selection of $W_n$ must be in

$$(21) \qquad TW_n - TS_n = F_{2n+2} - F_{2n-2} - F_{2n} - 2$$

trees.

## 7. GRAPHS WITH MULTIPLE SPOKES AND RIM EDGES

TREE COUNT WITH MULTIPLE SPOKES. Suppose that the number of spokes in position $h$ of $S_n$ is increased to $j$. Since a spoke cannot be in a tree with any other spoke in the same position (resulting loop could not be part of a tree), the number of trees would be (with the aid of (13) and (17)),

$$(22) \qquad TS_n + (j-1)T_{h-1}(1(2)) \cdot T_{n-h}(1(2)) = F_{2n} + (j-1)F_{2h-1} \cdot F_{2n-2h+1}, \qquad (1 \leqslant h \leqslant n).$$

Correspondingly, the increase of the number of spokes in any position of $W_{n+1}$ results in a total of trees given by (see (16) and (18))

$$(23) \qquad TW_n + (j-1)TS_{n-1}(1(2), n-1(2)) = F_{2n+2} - F_{2n-2} + (j-1)F_{2n} - 2 .$$

TREE COUNT WITH MULTIPLE RIM EDGES. If the number of rim edges for sector $k$ of $S_n$ is increased to $j$, the number of trees would become (with the aid of (13) and (20)),

$$(24) \qquad jTS_n - (j-1)TS_k \cdot TS_{n-k} = jF_{2n} - (j-1)F_{2k} \cdot F_{2n-2k} .$$

Also, if the number of rim edges for any given sector of $W_{n+1}$ were increased to $j$, the number of trees becomes (see (16) and (21))

$$(25) \qquad jTW_{n+1} - (j-1)TS_n = j(F_{2n+2} - F_{2n-2} - F_{2n} - 2) + F_{2n} .$$

Extensions to additional multiple edges are available only for the trying. One obvious use for the tree count formulas is the evaluation in general Fibonacci terms special determinants whose form fits $det\,(A_nA_n^t)$ for the multiple edge-modified sector graph or wheel.

Examples showing numbers of trees containing various edges are given in Figure 2.

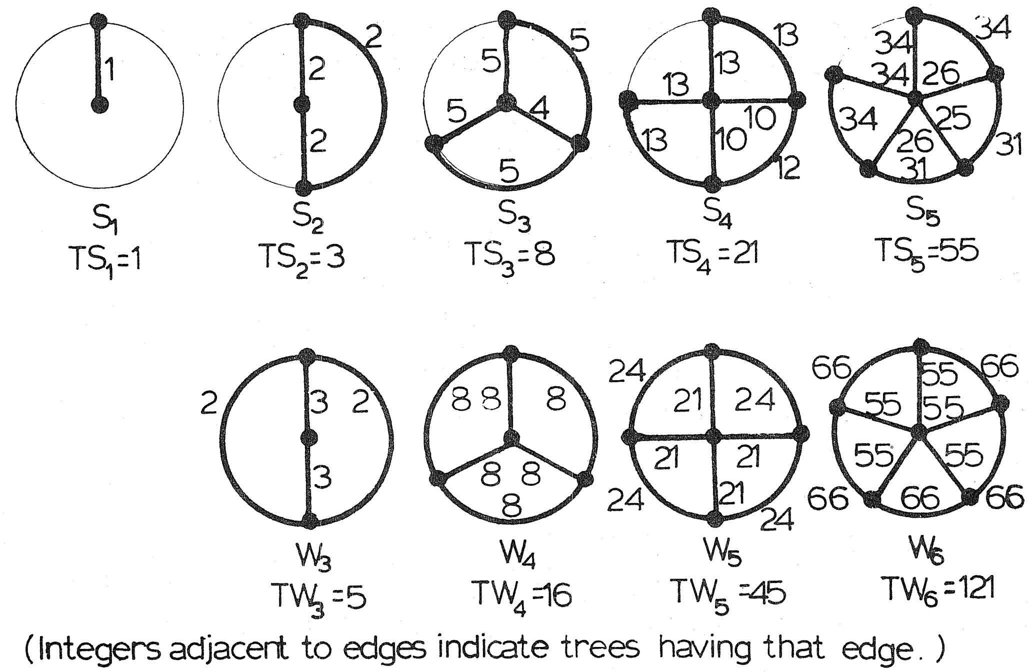


Figure 2. Examples of $S_n$ and $W_{n+1}$

## 8. SOME OBSERVATIONS

From Figure 2, it can be surmised that the sum of the number of trees containing edge one, edge two, etc., of $W_{n+1}$ is exactly $n$ times $TW_{n+1}$. Since there are $n$ spokes and $n$ rim edges in $W_{n+1}$, multiplication of the sum of (18) and (21) by $n$ does yield $n$ times (16), which verifies the surmise.

Also, from Figure 2 it can be surmised that $S_n$ has this same property. The surmise again is true and rests eventually on the identity

$$F_{2n-2} = \sum_{h=1}^{n-1} [F_{2h-1} \cdot F_{2n-2h+1} - F_{2h} \cdot F_{2n-2h}] \tag{26}$$

which is left as a search or research exercise for the reader.

## 9. REFERENCES

1. Harary, F., *Graph Theory,* Addison-Wesley Publishing Co., Reading, Mass., 1969, pp. 45–46.
2. R.G. Busacker and T.L. Saaty, *Finite Graphs and Networks,* McGraw-Hill Book Co., New York, N.Y., 1965, pp. 137–139.
3. S.L. Basin, Proposed problem B-13, *The Fibonacci Quarterly,* Vol. 1, No. 2 (April 1963), p. 86.
4. S.L. Basin, "The Fibonacci Sequence as it Appears in Nature," *The Fibonacci Quarterly,* Vol. 1, No. 1, (Feb. 1963), pp. 53–56.
5. D.K. Cheng, *Analysis of Linear Systems,* Addison-Wesley Publishing Co., Reading, Mass., 1959, pp. 313–320.
6. S.L. Hakimi, "On the Realizibility of a Set of Trees," *IRE Trans. Circuit Theory,* CT-8, 1961, pp. 11–17.

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