ON THE PARTITION OF HORADAM'S GENERALIZED SEQUENCES INTO GENERALIZED FIBONACCI AND GENERALIZED LUCAS SEQUENCES

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1. INTRODUCTION

If p,q are integers, $p^2+4q \neq 0$, let $\omega = \omega(p,q)$ be the set of those second-order integer sequences

$$(W_n) = (W_0, W_1, W_2, \cdots)$$

satisfying the relationship

$$W_n = pW_{n-1} + qW_{n-2} \quad (n \geqslant 2)$$

which are not also first-order sequences; i.e., they do not satisfy $W_n = cW_{n-1}\ (\forall n)$ for some c. In Horadam's papers ([3], [4], [5], [6]) our W_n is denoted by $W_n(a,b;p,-q)$. In this paper we show that ω can be partitioned naturally into a set $\mathcal{F}$ of generalized Fibonacci sequences and a set $\mathcal{L}$ of generalized Lucas sequences; to each $F \in \mathcal{F}$ there corresponds one $L \in \mathcal{L}$ and vice-versa. We also indicate how very many of the well-known identities may be generalized in a simple way.

2. THE PARTITION OF $\omega(p,q)$

If α,β are the roots of $x^2 - px - q = 0$, $d = +\sqrt{p^2+4q}$ then the following relationships are true:

$$\alpha = \frac{p+d}{2}, \qquad \beta = \frac{p-d}{2},$$

$$\alpha+\beta = p, \qquad \alpha\beta = -q, \qquad \alpha-\beta = d,$$

(1) $$W_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta},$$

where $A = W_1 - W_0\beta$, $B = W_1 - W_0\alpha$. Since (W_n) is not a first-order sequence it follows that $\alpha \neq 0, \beta \neq 0, A \neq 0, B \neq 0$. When W_n is represented as in (1) we say that W_n is in *Fibonacci form.* On the other hand, with different constants C and D, W_n could be represented as

$$W_n = C\alpha^n + D\beta^n .$$

In this case, we say that W_n is in *Lucas form.*

When W_n is in Fibonacci form (1) we may perform an operation (′) to obtain a number W'_n, where

$$W'_n = A\alpha^n + B\beta^n .$$

We say that the sequence (W'_n) is *derived from* the sequence (W_n). The sequence (W'_n) is a sequence of integers since

(2) $$W'_0 = A+B = W_1 - W_0\beta + W_1 - W_0\alpha = 2W_1 - W_0(\alpha+\beta) = 2W_1 - pW_0$$

and

(3) $$W'_1 = A\alpha + B\beta = (W_1 - W_0\beta)\alpha + (W_1 - W_0\alpha)\beta = W_1(\alpha+\beta) - 2W_0\alpha\beta = pW_1 + 2qW_0 .$$

W'_n may now be expressed in Fibonacci form. In that case

$$W'_n = \frac{[A(\alpha-\beta)]\,\alpha^n - [-B(\alpha-\beta)]\,\beta^n}{\alpha-\beta} .$$

If we perform the operation (′) on W'_n we obtain

$$\begin{aligned} W''_n &= [A(\alpha-\beta)]\,\alpha^n + [(-B)(\alpha-\beta)]\,\beta^n \\ &= (\alpha-\beta)^2 \left[\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right] \\ &= d^2 W_n . \end{aligned}$$

We have proved

Theorem 1. $W_n'' = d^2 W_n$ for all $n \geqslant 0$.

It is not hard to verify that the equation $W_n = W_n' \ (\forall_n)$ cannot be true if (W_n) is not a first-order sequence.

Throughout this paper let $(X_n), (Y_n) \in \omega(p,q)$, let $X_n' = Y_n \ (n = 0, 1, 2, \cdots)$ and let $X_0 = a$, $X_1 = b$. Then, from (2) and (3),

$$Y_0 = 2b - ap, \qquad Y_1 = pb + 2qa.$$

By theorem 1, therefore, or directly, it follows that

$$ad^2 = 2Y_1 - pY_0, \qquad bd^2 = pY_1 + 2qY_0.$$

The following theorem now follows easily:

Theorem 2. (i)

(4) $$d^2 \mid 2Y_n - pY_{n-1} \quad \text{and} \quad d^2 \mid pY_n + 2qY_{n-1} \quad \text{for all} \quad n \geqslant 1.$$

(ii) If $(W_n) \in \omega(p,q)$, $d^2 \mid 2W_1 - pW_0$ and $d^2 \mid pW_1 + 2qW_0$ then $(W_n) = (X_n')$ for some $(X_n) \in \omega(p,q)$.

Proof of (ii). If

$$X_0 = \frac{2W_1 - pW_0}{d^2}, \quad X_1 = \frac{pW_1 + 2qW_0}{d^2} \quad \text{and} \quad (X_n) \in \omega(p,q),$$

then

$$X_0' = 2\left(\frac{pW_1 + 2qW_0}{d^2}\right) - p\left(\frac{2W_1 - pW_0}{d^2}\right) = W_0 \quad \text{and} \quad X_1' = p\left(\frac{pW_1 + 2qW_0}{d^2}\right) + 2q\left(\frac{2W_1 - pW_0}{d^2}\right) = W_1$$

which proves part (ii).

The basic linear relationships connecting (X_n) and (Y_n) are described in the following theorem.

Theorem 3. The following are equivalent:

(i) $$(X_n') = (Y_n),$$

(ii) $$Y_n = 2X_{n+1} - pX_n \quad \text{for all} \quad n \geqslant 0,$$

(iii) $$Y_{n+1} = pX_{n+1} + 2qX_n \quad \text{for all} \quad n \geqslant 0,$$

(iv) $$Y_n = X_{n+1} + qX_{n-1} \quad \text{for all} \quad n \geqslant 1,$$

(v) $$X_n = \frac{2Y_{n+1} - pY_n}{d^2} \quad \text{for all} \quad n \geqslant 0,$$

(vi) $$X_{n+1} = \frac{pY_{n+1} + 2qY_n}{d^2} \quad \text{for all} \quad n \geqslant 0,$$

(vii) $$X_n = \frac{Y_{n+1} + qY_{n-1}}{d^2} \quad \text{for all} \quad n \geqslant 1.$$

NOTE: For each of (ii), $\cdots$, (vii) we need only require that the expression is true for two adjacent values of n.

Proof. (i) $\Rightarrow$ (ii). If $(X_n') = (Y_n)$, then from (2) and (3), $Y_0 = 2X_1 - pX_0$ and $Y_1 = pX_1 + q2X_0 = 2X_2 - pX_1$ since $X_2 = pX_1 + qX_0$. Let $m \geqslant 2$ and assume (ii) is true for $0 \leqslant n < m$. Then

$$Y_m = pY_{m-1} + qY_{m-2} = p(2X_m - pX_{m-1}) + q(2X_{m-1} - pX_{m-2}) = 2X_{m+1} - pX_m.$$

The result now follows by induction.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow \cdots \Leftrightarrow$ (vii). This follows easily using

$$X_{n+1} = pX_n + qX_{n-1} \quad \text{and} \quad Y_{n+1} = pY_n + qY_{n-1} \quad (n \geqslant 1).$$

[(ii), (iii), $\cdots$ (vii)] $\Rightarrow$ (i). Since

$$X_0 = \frac{2Y_1 - pY_0}{d^2} \quad \text{and} \quad X_1 = \frac{pY_1 + 2qY_0}{d^2}$$

it follows from (2) and (3) that

$$X_0' = 2\left(\frac{pY_1 + 2qY_0}{d^2}\right) - p\left(\frac{2Y_1 - pY_0}{d^2}\right) = Y_0$$

and similarly $X_1' = Y_1$. Hence $(X_n') = (Y_n)$. This completes the proof of Theorem 3.

We now describe the partition of $\omega(p,q)$ previously referred to:

If $(W_n) \in \omega(p,q)$ and $d \neq 1$ let $W_n = d^{2m}\omega_n$ for all $n \geqslant 0$, where $m \geqslant 0$ is an integer, $(\omega_n) \in \omega$ and $d^2 \nmid \omega_n$ for at least one $n \geqslant 0$. Then

$$(W_n) \in L \quad \text{if} \quad d^2 | 2\omega_1 - p\omega_0 \quad \text{and} \quad d^2 | p\omega_1 + 2q\omega_0 ,$$
$$(W_n) \in F \quad \text{if either} \quad d^2 \nmid 2\omega_1 - p\omega_0 \quad \text{or} \quad d^2 \nmid p\omega_1 + 2q\omega_0 .$$

If $(W_n) \in \omega(p,q)$ and $d = 1$ let

$$(W_n) \in L \quad \text{if} \quad W_1 - W_0\alpha < 0,$$
$$(W_n) \in F \quad \text{if} \quad W_1 - W_0\alpha > 0.$$

The assignment of (W_n) to L or F is natural in the case $d \neq 1$, but if $d = 1$, although the partition itself is natural, it is not true to say that a sequence is "like" the Lucas sequence rather than the Fibonacci sequence or vice-versa. In view of Theorem 3 if (W_n) is a member of F (or L) then any "tail" of (W_n) is also a member of F (or L, respectively).

Theorem 4. $(X_n) \in F$ if and only if $(Y_n) \in L$.

Proof. Case 1. $d = 1$. $(X_n) \in F$

$$\Leftrightarrow X_n = A\alpha^n - B\beta^n , \quad \text{where} \quad B < 0$$
$$\Leftrightarrow Y_n = A\alpha^n + B\beta^n$$
$$\Leftrightarrow (Y_n) \in L .$$

Case 2. $d \neq 1$. (i) If $(X_n) \in F$ suppose that $X_n = d^{2m}x_n$ for all $n \geqslant 0$, where $m \geqslant 0$ is an integer, $(x_n) \in F$ and $d^2 \nmid x_n$ for at least one $n \geqslant 0$. Clearly $d^2 \nmid x_0$ or $d^2 \nmid x_1$. By Theorem 3, $Y_0 = 2X_1 - pX_0$ and $Y_1 = pX_1 + 2qX_0$. Let $Y_n = d^{2m}y_n$ for all $n \geqslant 0$. Then $y_0 = 2x_1 - px_0$ and $y_1 = px_1 + 2qx_0$. Since $(x_n) \in F$, either $d^2 \nmid 2x_1 - px_0$ or $d^2 \nmid px_1 + 2qx_0$. Therefore either $d^2 \nmid y_0$ or $d^2 \nmid y_1$. But it is easy to verify that

$$2y_1 - py_0 = d^2x_0 \quad \text{and} \quad py_1 + 2qy_0 = d^2x_1 .$$

Therefore $(y_n) \in L$ and so $(Y_n) \in L$.

(ii) If $(Y_n) \in L$ suppose that $Y_n = d^{2m}y_n$ for all $n \geqslant 0$, where $m \geqslant 0$ is an integer, $(y_n) \in L$ and $d^2 \nmid y_n$ for at least one $n \geqslant 0$. Clearly $d^2 \nmid y_0$ or $d^2 \nmid y_1$. By Theorem 3,

$$X_0 = \frac{2Y_1 - pY_0}{d^2}, \qquad X_1 = \frac{pY_1 + 2qY_0}{d^2} .$$

Let $X_n = d^{2m}x_n$ for all $n \geqslant 0$. Then

$$x_0 = \frac{2y_1 - py_0}{d^2}, \qquad x_1 = \frac{py_1 + 2qy_0}{d^2} .$$

Since $(y_n) \in L$,

$$d^2 | 2y_1 - py_0 \quad \text{and} \quad d^2 | py_1 + 2qy_0 ,$$

so x_0 and x_1 are integers, so $(x_n) \in \omega$. But

$$2x_1 - px_0 = y_0 \quad \text{and} \quad px_1 + 2qx_0 = y_1 ,$$

and since $d^2 \nmid y_0$ or $d^2 \nmid y_1$ it follows that either $d^2 \nmid 2x_1 - px_0$ or $d^2 \nmid px_1 + 2qx_0$. Therefore $(x_n) \in F$ and so $(X_n) \in F$. This completes the proof of Theorem 4.

Here are some examples of members of F alongside the corresponding member of L.

$0, 1, 1, 2, 3, 5, 8, 13, \cdots$	$2, 1, 3, 4, 7, 13, \cdots$
$0, 1, p, p^2 + q, \cdots$	$2, p, p^2 + 2q, \cdots$
$0, 1, 3, 7, 15, \cdots, 2^n - 1, \cdots$	$2, 3, 5, 9, 17, \cdots, 2^n + 1, \cdots$
$0, 1, 2, 5, 12, 29, \cdots$	$2, 2, 6, 14, \cdots$
(Pell's sequences)	
$a, b, qa, qb, q^2a, q^2b, \cdots$	$2b, 2qa, 2qb, 2q^2a, 2q^2b, \cdots$

3. BINOMIAL IDENTITIES

Many identities involving Fibonacci and Lucas numbers are readily derived from the binomial theorem; for example see [1], [2] or [8]. They can nearly always be generalized to become identities involving generalized Fibonacci and Lucas numbers.

In this section we could derive a long list of such identities; but this seems unnecessary in view of the proofs in [2] and [8], and also it would take up a lot of space, as the constant multipliers which have to be introduced seem to make the generalized formulae up to twice as long as the formulae in [2] and [8]. Instead we derive one set of identities as an example and show how further identities may be derived.

There often seem to be two very similar identities, one featuring Fibonacci numbers, the other Lucas numbers. When there are two such identities they may often be derived from one identity by using the fact that 1 and $\sqrt{5}$ are linearly independent over the rationals, although this is not the procedure adopted in [2] or [8]. With generalized Fibonacci and Lucas numbers such a process would not be appropriate, but, as the examples show, the method of proof which is natural does lead to a single identity, from which the two identities may be obtained by specialization.

For this section (F_n) and (L_n) denote a pair of sequences such that $(F_n) \in F$, $(L_n) \in L$ and $(F_n)' = (L_n)$. Also, $C = F_1 - F_0\beta$, $D = F_1 - F_0\alpha$.

The natural method of proof is firstly to derive a single identity involving (X_n) and (Y_n). Then either of the following sets of substitutions may be made:

I.
$$X_n = F_{n+r}$$
$$Y_n = L_{n+r}$$
$$A = X_1 - X_0\beta = F_{r+1} - F_r\beta = C\alpha^r$$
$$B = X_1 - X_0\alpha = F_{r+1} - F_r\alpha = D\beta^r .$$

(The third of these follows since

$$F_{r+1} = \frac{C\alpha^{r+1} - D\beta^{r+1}}{\alpha-\beta} = \frac{(C\alpha^r - D\beta^r)\beta - C\alpha^r\beta + C\alpha^{r+1}}{\alpha-\beta} = F_r\beta + C\alpha^r ,$$

and the fourth follows similarly.)

Or

II.
$$X_n = L_{n+r}$$
$$Y_n = d^2F_{n+r}$$
$$A = X_1 - X_0\beta = L_{r+1} - L_r\beta = Cd\alpha^r$$
$$B = X_1 - X_0\alpha = L_{r+1} - L_r\alpha = -Dd\beta^r .$$

Then each of these sets of substitutions leads to one of the two derived identities mentioned above.

EXAMPLES OF BINOMIAL IDENTITIES

EXAMPLE 1. Since

$$\alpha^m = \frac{Y_m + dX_m}{2A}, \qquad \beta^m = \frac{Y_m - dX_m}{2B}$$

it follows that

$$\alpha^{mn} = (2A)^{-n}\sum_{i=0}^{n} d^iX_m^iY_m^{n-i}\binom{n}{i}, \quad \text{and} \quad \beta^{mn} = (2B)^{-n}\sum_{i=0}^{n}(-1)^id^iX_m^iY_m^{n-i}\binom{n}{i} .$$

Therefore,

$$Y_{mn} + dX_{mn} = (2A)^{1-n}\sum_{i=0}^{n} d^iX_m^iY_m^{n-i}\binom{n}{i}, \quad \text{and} \quad Y_{mn} - dX_{mn} = (2B)^{1-n}\sum_{i=0}^{n}(-1)^id^iX_m^iY_m^{n-i}\binom{n}{i}.$$

Therefore,

$$X_{mn} = 2^{-n}d^{-1}\sum_{i=0}^{n}(dX_m)^iY_m^{n-i}\binom{n}{i}\,[A^{1-n} - (-1)^iB^{1-n}] .$$

A similar formula may be derived for Y_{mn}.

Making the first set of substitutions, we obtain

$$F_{mn+r} = 2^{-n}d^{-1}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}([C\alpha^r]^{1-n} - (-1)^i[D\beta^r]^{1-n}) .$$

But

$$C^{1-n}\alpha^{r-rn} - (-1)^i D^{1-n}\beta^{r-rn} = C^{1-n}\left(\frac{L_{r-rn}+dF_{r-rn}}{2C}\right) - \frac{(-1)^i D^{1-n}(L_{r-rn}-dF_{r-rn})}{2D}$$

$$= \frac{L_{r-rn}}{2}\left(\frac{1}{C^r} - (-1)^i\frac{1}{D^r}\right) + \frac{dF_{r-rn}}{2}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right).$$

Therefore

$$F_{nm+r} = 2^{-n-1}d^{-1}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}\left\{L_{r-rn}\left(\frac{1}{C^n} - (-1)^i\frac{1}{D^n}\right) + dF_{r-rn}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right)\right\}.$$

Making the second set of substitutions we obtain

$$L_{nm+r} = 2^{-n}d^{-1}\sum_{i=0}^{n}(dL_{m+r})^i(d^2F_{m+r})^{n-i}\binom{n}{i}([Cd\alpha^r]^{1-n} - (-1)^i[-Dd\beta^r]^{1-n})$$

$$= 2^{-n}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}([C\alpha^r]^{1-n} - (-1)^{n-i}(-1)^{1-n}[D\beta^r]^{1-n})$$

$$= 2^{-n}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}([C\alpha^r]^{1-n} + (-1)^i[D\beta^r]^{1-n}).$$

But

$$C^{1-n}\alpha^{r-rn} + (-1)^i D^{1-n}\beta^{r-rn} = \frac{L_{r-rn}}{2}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right) + dF_{r-rn}\left(\frac{1}{C^n} - (-1)^i\frac{1}{D^n}\right) .$$

Therefore

$$L_{mn+r} = 2^{-n-1}\sum_{i=0}^{n}(dF_{m+r})^i L_{m+r}^{n-i}\binom{n}{i}\left\{L_{r-rn}\left(\frac{1}{C^n} + (-1)^i\frac{1}{D^n}\right) + dF_{r-rn}\left(\frac{1}{C^n} - (-1)^i\frac{1}{D^n}\right)\right\}.$$

EXAMPLE 2. Since

$$dX_m = 2A\alpha^m - Y_m \qquad \text{and} \qquad dX_m = -(2B\beta^m - Y_m)$$

it follows that

$$\alpha^k d^n X_m^n = \sum_{i=0}^{n}(-Y_m)^i(2A)^{n-i}\binom{n}{i}\alpha^{mn-mi+k} \quad\text{and}\quad \beta^k d^n X_m^n = (-1)^n\sum_{i=0}^{n}(-Y_m)^i(2B)^{n-i}\binom{n}{i}\beta^{mn-mi+k}.$$

Therefore

$$Y_k d^n X_m^n + X_k d^{n+1}X_m^n = \sum_{i=0}^{n}(-Y_m)^i(2A)^{n-i}\binom{n}{i}(Y_{mn-mi+k} + dX_{mn-mi+k})$$

and

$$Y_k d^n X_m^n - X_k d^{n+1}X_m^n = (-1)^n\sum_{i=0}^{n}(-Y_m)^i(2\beta)^{n-i}\binom{n}{i}(Y_{mn-mi+k} - dX_{mn-mi+k}) .$$

Therefore

$$X_kX_m^n = \frac{1}{2d^{n+1}}\sum_{i=0}^{n}(-1)^iY_m^i2^{n-i}\binom{n}{i}[Y_{mn-mi+k}(A^{n-i}-(-1)^n\beta^{n-i})+dX_{mn-mi+k}(A^{n-i}+(-1)^n\beta^{n-i})]$$

and

$$Y_kX_m^n = \frac{1}{2d^n}\sum_{i=0}^{n}(-1)^iY_m^i2^{n-i}\binom{n}{i}[Y_{mn-mi+k}(A^{n-i}+(-1)^n\beta^{n-i})+dX_{mn-mi+k}(A^{n-i}-(-1)^n\beta^{n-i})].$$

Making the first set of substitutions we obtain

$$F_kF_m^n = \frac{1}{2d^{n+1}}\sum_{i=0}^{n}(-1)^iL_m^i2^{n-i}\binom{n}{i}[L_{mn-mi+k}(C^{n-i}-(-1)^nD^{n-i})+dF_{mn-mi+k}(C^{n-i}+(-1)^nD^{n-i})]$$

and

$$L_kF_m^n = \frac{1}{2d^n}\sum_{i=0}^{n}(-1)^iL_m^i2^{n-i}\binom{n}{i}[L_{mn-mi+k}(C^{n-i}+(-1)^nD^{n-i})+dF_{mn-mi+k}(C^{n-i}-(-1)^nD^{n-i})].$$

Making the second set of substitutions we obtain

$$d^2F_kL_m^n = \frac{1}{2d^n}\sum_{i=0}^{n}(-1)^i(d^2F_m)^i2^{n-i}\binom{n}{i}[d^2F_{mn-mi+k}(d^{n-i}C^{n-i}+(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})$$
$$+dL_{mn-mi+k}(d^{n-i}C^{n-i}-(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})]$$

so that

$$F_kL_m^n = \frac{1}{2d}\sum_{i=0}^{n}(dF_m)^i2^{n-i}\binom{n}{i}[dF_{mn-mi+k}(D^{n-i}+(-1)^iC^{n-i})-L_{mn-mi+k}(D^{n-i}-(-1)^iC^{n-i})]$$

and

$$L_kL_m^n = \frac{1}{2d^{n+1}}\sum_{i=0}^{n}(-1)^i(d^2F_m)^i2^{n-i}\binom{n}{i}[d^2F_{mn-mi+k}(d^{n-i}C^{n-i}-(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})$$
$$+dL_{mn-mi+k}(d^{n-i}C^{n-i}+(-1)^n(-1)^{n-i}d^{n-i}D^{n-i})]$$

so that

$$L_kL_m^n = \frac{1}{2}\sum_{i=0}^{n}(dF_m)^i2^{n-i}\binom{n}{i}[L_{mn-mi+k}(D^{n-i}+(-1)^iC^{n-i})-dF_{mn-mi+k}(D^{n-i}-(-1)^iC^{n-i})].$$

Further three term identities from which binomial identities may be derived in the way described are

$$dX_n = A\alpha^n - B\beta^n ,$$

$$Y_n = A\alpha^n + B\beta^n ,$$

(5) $$A\alpha^{m+n} = X_m\alpha^{n+1} + qX_{m-1}\alpha^n ,$$

(6) $$B\beta^{m+n} = X_m\beta^{n+1} + qX_{m-1}\beta^n ,$$

$$\alpha^2 = p\alpha + q ,$$

$$\beta^2 = p\beta + q ,$$

(7) $$Y_n^2 = d^2X_n^2 + 4AB(-q)^j ,$$

$$A\alpha^{2m} = Y_m\alpha^m - B(-q)^m ,$$

$$A\alpha^{2m} = dX_m\alpha^m + B(-q)^m$$

$$B\beta^{2m} = Y_m\beta^m - A(-q)^m , \qquad B\beta^{2m} = -dX_m\beta^m + A(-q)^m .$$

Most of these identities are obvious, or nearly so. Identity (5) may be proved as follows:

$$A\alpha^m = \tfrac{1}{2}Y_m + \tfrac{1}{2}dX_m = \tfrac{1}{2}(pX_m + 2qX_{m-1} + dX_m) = X_m\left(\frac{p+d}{2}\right) + qX_{m-1} = X_m\alpha + qX_{m-1}\,,$$

and identity (6) is proved similarly. Identity (7) is proved as follows:

$$Y_n^2 = (A\alpha^n + B\beta^n)^2 = (A\alpha^n - B\beta^n) + 4AB(\alpha\beta)^n = (\alpha-\beta)^2\left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right)^2 + 4AB(-q)^n = d^2X_n^2 + 4AB(-q)^n.$$

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ERRATA

Please make the following corrections to "Fibonacci Sequences Modulo *M*," appearing in the February 1974 (Vol. 12, No. 1) issue of *The Fibonacci Quarterly,* pp. 51–64.

On page 52, last line, last sentence, change "If $2/f(p)$," to read "If $2\nmid f(p)$."

On page 53, change the fourth line of the third paragraph from "which $(a,b,p^e) = 1$," to: "which $(a,b,p^e) \neq 1$."

On page 56, third paragraph of proof, tenth line should read:

"...is given by $5^{2e} - 5^{2e-2} - 4\cdot 5^{2e-2} = 4\cdot 5^{2e-1}$ ... ."

On page 61, change the second displayed equation to read:

$$n(k) = \frac{p^{2t}-1}{k}\,.$$

Line 7 from the bottom should read:

" for $i = t, \cdots, e-1.$ "