P·Q M-CYCLES, A GENERALIZED NUMBER PROBLEM

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In this note all letters will denote non-negative integers. A number

$$N = n_1 \cdot 10^k + n_2 \cdot 10^{k-1} + \cdots + n_{k-1} \cdot 10 + n_k$$

(abbreviated $N = n_1 n_2 \cdots n_k$) will be called a $p \cdot q$ m-cycle whenever

$$p(n_{k-m-1}, n_{k-m-2}, \cdots, n_{k-1} n_k n_1 \cdots n_{k-m}) = q(n_1 n_2 \cdots n_k).$$

Since four parameters $\{p,q,m,k\}$ are involved, some rather interesting questions and conjectures arise naturally. The problem of Trigg [3], for example, yielded 428571, a distinct (i.e., the digits are distinct) $3 \cdot 4$ 3-cycle when $k = 6$, and $1 \cdot 1$ m-cycles which are n-linked were considered in [2]. Klamkin [1] recently characterized the smallest $1 \cdot 6$ 1-cycles. Here we extend some of these concepts, show how to generate various $p \cdot q$ m-cycles, and actually produce the smallest $1 \cdot q$ 1-cycles $(q = 1, 2, \cdots, 9)$ together with some of their properties. As a special case of our more generalized results, we present a much faster method than Wlodarski [4] for obtaining the smallest $1 \cdot q$ 1-cycles with $n_k = q$.

For notation, $n_1 \cdot n_2$ means n_1 times n_2, whereas $n_1 n_2$ will denote the two-digit number $10n_1 + n_2$. For a number $r \cdot s = n_1 n_2$, we shall use $(r \cdot s)_{10} = n_1$ and $(r \cdot s)_1 = n_2$.

1. $1 \cdot q$ 1-CYCLES

We first note that for each q $(q = 1, 2, \cdots, 9)$ and each $n_1 \leqslant 9/q$, there exists a smallest (unique non-repeating) $1 \cdot q$ 1-cycle

$$N_q(n_1) = n_1 n_2 \cdots n_{k_q(n_1)}$$

($k_q(n_1)$, the number of digits in $N_q(n_1)$ will depend on q and n_1). Indeed, assume that $k_q(n_1)$ is not fixed and note that $n_{k_q(n_1)} = q \cdot n_1 \neq 0$ when $n_1 \neq 0$. Then $N_q(n_1)$ is readily obtained by the following simple multiplication:

$$\begin{array}{lcccc}
 & n_1 & n_{k-2} & n_{k-1} & n_k \\
N = & n \quad \cdots & [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 & (q \cdot n_k)_1 & q \cdot n \\
qN = & q \cdot n \cdots & \{q \cdot n_{k-2} + [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_{10}\}_1 & [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 & (q \cdot n_k)_1
\end{array}$$

EXAMPLE 1. 025641 and 205128 are $1 \cdot 4$ 1-cycles, whereas 142857 is a $1 \cdot 5$ 1-cycle. These numbers were obtained from

$$\begin{array}{llll}
 n_1 & n_{k_4(2)} & n_1 & n_{k_5(1)} \\
N = 2\,0\,5\,1\,2 & 8 & N = 1\,4\,2\,8\,5 & 7 \\
4N = 8\,2\,0\,5\,1 & 2 = (4 \cdot 8)_1 & 3N = 7\,1\,4\,2\,8 & 5 = (5 \cdot 7)_1
\end{array}$$

For $n_1 = 1$, the above procedure yields the following $1 \cdot q$ 1-cycles $N_q(1)$. (Note that by simply placing $n_1 = 1$ after $n_{qk(1)}$, one obtains the corresponding $1 \cdot q$ 1-cycles $N_q(0) = 0n_3 n_4 \cdots q1$).

q	$N_q(1)$	$k_q(1)$
1	u,u where $u = 0, 1, 2, \cdots, 9$	2
2	105263157894736842	18
3	1034482758620689655172413793	28
4	102564	6
5	102040816326530612244897959183673469387755	42
6	1016949152542372881355932203389830508474576271186440677966	58
7	1014492753623188405797	22
8	1012658227848	13
9	10112359550561797752808988764044943820224719	44

We note here that there does not exist a largest $1 \cdot q$ 1-cycle $N_q(n_1) > 1$ since $n_1 n_2 \cdots n_k n_1 n_2 \cdots n_k$ is a $1 \cdot q$ 1-cycle for each $1 \cdot q$ 1-cycle $N_q(n_1)$.

EXAMPLE 2. The smallest (nonzero) $1 \cdot q$ 1-cycles are given by

$$N_1(1), \quad N_q(0) \quad \text{for } q = 2, 3, 4, 5, 6, 7, 8, 9$$

Indeed,

$$N_2(4) > N_2(3) > N_2(2) > N_2(1)$$

$$N_3(3) > N_3(2) \qquad \text{and} \qquad N_4(2) = 205128 > N_4(1) > N_4(0).$$

For $q \geqslant 5$, the only nonrepeating $1 \cdot q$ 1-cycles are $N_q(0)$ and $N_q(1)$.

We conclude this section by mentioning that the smallest $1 \cdot q$ 1-cycles whose last term $n_{k_q(n_1)} = q$ are precisely the numbers $N_q(1)$ in the above table.

2. $p \cdot q$ 1-CYCLES

Each $1 \cdot q$ 1-cycle is a $p \cdot p \cdot q$ 1-cycle for every integer p, and every $p \cdot q$ 1-cycle is clearly a

$$\frac{p}{(p,q)} \cdot \frac{q}{(p,q)}$$

1-cycle. To obtain $p \cdot q$ 1-cycles $N = n_1 n_2 \cdots n_k$ in general, let

$$N' = n_k n_1 \cdots n_{k-1}.$$

Then $pN' = qN$ requires that $n_k \leqslant n_1$ when $p > q$ and $n_k \geqslant n_1$ for $p < q$, and since

$$(p \cdot n_{k-1})_1 = (q \cdot n_k)_1,$$

we use n_k as a sieve for a generalization of the multiplication given in Section 1. Thus, keeping

$$(p \cdot n_{k-1})_1 = (q \cdot n_k)_1, \quad [p \cdot n_{k-2} + (p \cdot n_{k-1})_{10}]_1 = [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1,$$

etc., we proceed until the m^{th} position (denoted by a vertical line preceeding the $n_{k-m}{}^{th}$ digit of N), where the sequence of digits begin to repeat anew in the $m + 1^{st}$ position.

$$\begin{array}{rcl|ccc}
N' = & n_k & & & n_{k-2} & n_{k-1} \\
pN' = & \cdots\cdots\cdots\cdot & & \cdot\,\cdot & [p \cdot n_{k-2} + (p \cdot n_{k-1})_{10}]_1 & (p \cdot n_{k-1})_1 \\
qN = & \cdots\cdots\cdots\cdot & & \cdot\,\cdot & [q \cdot n_{k-1} + (q \cdot n_k)_{10}]_1 & (q \cdot n_k)_1 \\
N = & n_1 & & & n_{k-1} & n_k
\end{array}$$

EXAMPLE 3. (i) 162 is a $3 \cdot 4$ 1-cycle.

(ii) 21 is a $7 \cdot 4$ 1-cycle.

(iii) There does not exist a $5 \cdot 8$ 1-cycle.

(i) Since

$$(3 \cdot n_{k-1})_1 = (4 \cdot n_k)_1,$$

and since $n_k > n_1$, consider $n_k = 2$ (therefore $n_{k-1} = 6$). Then the above multiplication yields

$$\begin{array}{rl} N' = & 2\ \ 1\ n_2\ n_3 \cdots n_{k-3}\ n_{k-2}\ 6 \\ 3N' = & \phantom{2\ 1\ n_2\ n_3 \cdots n_{k-3}\ n_{k-2}}\ 8 \\ 4N = & \phantom{2\ 1\ n_2\ n_3 \cdots n_{k-3}\ n_{k-2}}\ 4\ 8 \\ N = & 1\ n_2\ n_3\ n_4\ \ \ n_{k-2}\ \ 6\ \ 2 \end{array}$$

Since

$$(3n_{k-2} + 1)_1 = 4 \Rightarrow n_{k-2} = 1$$

and

$$(3n_{k-3})_1 = 6 \Rightarrow n_{k-3} = 2\,,$$

$$\begin{array}{rl|l} 3N' = & 3(2\ \ 1\ n_3\ n_4 \cdots & 2\ 1\ 6) \\ & & 6\ 4\ 8 \\ & & 6\ 4\ 8 \\ 4N = & 4(1\ n_2\ n_3\ n_4 \cdots & 1\ 6\ 2) \end{array}$$

from which it readily follows that 162 is a $3 \cdot 4$ 1-cycle.

(ii) $(7n_{k-1})_1 = (4n_k)_1$ is satisfied by the pairs n_{k-1}, n_k

$$\begin{array}{c|ccccccccc} n_k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ n_{k-1} & 2 & 4 & 6 & 8 & 0 & 2 & 4 & 6 & 8 \end{array}\ .$$

Using the first pair yields

$$\begin{array}{r|l} 7(\ 1\ n_1\ n_2 \cdots & 1\ 2) \\ & 8\ 4 \\ & 8\ 4 \\ 4(n_1\ n_2\ n_3 \cdots & 2\ 1) \end{array}\ ,$$

The numbers $42, 63$ and 84 are also $7 \cdot 4$ 1-cycles.

(iii) None of the pairs of values satisfying

$$(5n_{k-1})_1 = (8n_k)_1$$

yield $5N' = 8N$.

3. P·Q M-CYCLES

The procedure of Section 2, appropriately modified, also applies to $p \cdot q$ m-cycles in general. We demonstrate this in

EXAMPLE 4. Find a distinct $3 \cdot 4$ 3-cycle for $k = 6$.

For

$$(4n_k)_1 = (3n_{k-3})_1$$

which is satisfied by numerous values, first consider $n_k = 1$ and $n_{k-3} = 8$. Then

$$\begin{array}{l} 3(n_{k-2}\ n_{k-1}\ \ 1\ n_1\ n_2 \cdots n_{k-6}\ n_{k-5}\ n_{k-4}\ 8) \\ \phantom{3(n_{k-2}\ n_{k-1}\ \ 1\ n_1\ n_2 \cdots n_{k-6}\ n_{k-5}\ n_{k-4}}\ 4 \\ \phantom{3(n_{k-2}\ n_{k-1}\ \ 1\ n_1\ n_2 \cdots n_{k-6}\ n_{k-5}\ n_{k-4}}\ 4 \\ 4(n_1\ \ n_2\ \ n_3\ n_4\ n_5 \cdots\ \ 8\ \ n_{k-2}\ n_{k-1}\ \ 1) \end{array}$$

yields

$$
\begin{array}{r|l}
3(n_{k-2}\ n_{k-1}\quad 1\ n_1\ n_2 \cdots & 5\ 7\ 1\ 4\ 2\ 8) \\
 & 7\ 1\ 4\ 2\ 8\ 4 \\
 & 7\ 1\ 4\ 2\ 8\ 4 \\
4(n_1 \qquad n_2\ n_3\ n_4\ n_5 \cdots & 4\ 2\ 8\ 5\ 7\ 1)
\end{array}
$$

so that *428571* is a solution to our problem.

REFERENCES

1. M.S. Klamkin, "A Number Problem," *The Fibonacci Quarterly,* Vol. 10, No. 3 (April 1972), p. 324.
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3. C.W. Trigg, "A Cryptarithm Problem," *Mathematics Magazine,* Vol. 45 (January 1972), p. 46.
4. J. Wlodarski, "A Number Problem" *The Fibonacci Quarterly,* Vol. 9 (April 1971), p. 195.

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THE APOLLONIUS PROBLEM

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Problem 29 on page 216 of E.W. Hobson's *A Treatise on Plane Trigonometry,"* Cambridge University Press (1918) reads: "Three circles, whose radii are $a, b, c,$ touch each other externally; prove that the radii of the two circles which can be drawn to touch the three are

$$abc/[(bc + ca + ab) \pm 2\sqrt{abc(a+b+c)}\,]."$$

Horner [1] states "The formula...is due to Col. Beard" [2]. That the formula is incorrect is evident upon putting $a = b = c,$ whereupon the radii become $a/(3 \pm 2\sqrt{3})$, so that one of them is negative. Horner recognized this when he stated, "The negative sign gives R (absolute value)...".

The correct formula has been shown [3] to be:

$$abc/[2\sqrt{abc(a+b+c)} \pm (ab + bc + ca)].$$

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3. C.W. Trigg, "Corrected Solution to Problem 2293, *School Science and Math.,* 53 (Jan. 1953), p. 75.

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