A FAREY SEQUENCE OF FIBONACCI NUMBERS

KRISHNASWAMI ALLADI
Vivekananda College, Madras–600004, India

The Farey sequence is an old and famous set of fractions associated with the integers. We here show that if we form a Farey sequence of Fibonacci Numbers, the properties of the Farey sequence are remarkably preserved (see [2]). In fact we find that with the new sequence we are able to observe and identify "points of symmetry," "intervals," "generating fractions" and "stages." The paper is divided into three parts. In Part 1, we define "points of symmetry," "intervals" and "generating fractions" and discuss general properties of the Farey sequence of Fibonacci numbers. In Part 2, we define conjugate fractions and deal with properties associated with intervals. Part 3 considers the Farey sequence of Fibonacci numbers as having been divided into stages and contains properties associated with "corresponding fractions" and "corresponding stages." A generalization of the Farey sequence of Fibonacci numbers is given at the end of the third part.

The Farey sequence of Fibonacci numbers of order F_n (where F_n stands for the n^{th} term of the Fibonacci sequence) is the set of all possible fractions F_i/F_j, $i = 0, 1, 2, 3, \cdots, n-1$, $j = 1, 2, 3, \cdots, n$ $(i < j)$ arranged in ascending order of magnitude. The last term is $1/1$, i.e., F_1/F_2. The first term is $0/F_{n-1}$. We set $F_0 = 0$ so that $F_0 + F_1 = F_2$, $F_1 = F_2 = 1$.

For convenience we denote a Farey sequence of Fibonacci numbers by $f \cdot f$, that of order F_n by $f \cdot f_n$ and the r^{th} term in the new Farey sequence of order F_n by $f_{(r)n}$.

PART 1

DEFINITION 1.1. Besides $1/1$ we define an $f_{(r)n}$ to be a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same denominator. We have shown in an appendix the Farey sequence of all Fibonacci numbers up to 34.

DEFINITION 1.2. We define an interval to be set of all $f \cdot f_n$ fractions between two consecutive points of symmetry. The interval may be closed or open depending upon the inclusion or omission of the points of symmetry. A closed interval is denoted by [ ] and an open interval by ( ).

DEFINITION 1.3. The distance between $f_{(r)k}$ and $f_{(k)n}$ is equal to $|r-k|$.

Theorem 1.1. If $f_{(r)n}$ is a point of symmetry then it is of the form $1/F_i$. Moreover $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same denominator if they do not pass beyond the next point of symmetry on either side. The converse is also true.

Proof. In the $f \cdot f$ sequence the terms are arranged in the following fashion. The terms in the last interval are of the form F_{j-1}/F_j. The terms in the interval prior to that last are of the form $F_{j-2}/F_j \cdots$. If there are two fractions F_{i-1}/F_{j-1} and F_{i-2}/F_{j-2} then their mediant* F_i/F_j lies in between them. That is,

$$\text{if} \quad \frac{F_{i-1}}{F_{j-1}} < \frac{F_{i-2}}{F_{j-2}} \qquad \text{then} \quad \frac{F_{i-1}}{F_{j-1}} < \frac{F_i}{F_j} < \frac{F_{i-2}}{F_{j-2}}$$

$$\text{if} \quad \frac{F_{i-2}}{F_{j-2}} < \frac{F_{i-1}}{F_{j-1}} \qquad \text{then} \quad \frac{F_{i-2}}{F_{j-2}} < \frac{F_i}{F_j} < \frac{F_{i-1}}{F_{j-1}} .$$

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*If $a/b < c/d$, then $(a+c)/(b+d)$ is the *mediant* fraction to those two fractions.

This inequality can easily be established dealing with the two cases separately.

We shall adopt induction as the method of proof. Our surmise has worked for all $f\cdot f$ sequences up to 34. Let us treat 34 as F_{n-1}. For the next $f\cdot f$ sequence, i.e., of order F_n, fractions to be introduced are:

$$\frac{F_2}{F_n}, \frac{F_3}{F_n}, \cdots, \frac{F_i}{F_n}, \cdots, \frac{F_{n-1}}{F_n}.$$

F_i/F_n will fall in between

$$\frac{F_{i-1}}{F_{n-1}} \quad \text{and} \quad \frac{F_{i-2}}{F_{n-2}}.$$

First assume that $F_{i-1}/F_{n-1} < F_{i-2}/F_{n-2}$. Since our assumption is valid for 34, F_{i-1}/F_{n-1} lies just before F_{i-2}/F_{n-2}. F_{i-3}/F_{n-2} will occur just after F_{i-2}/F_{n-1} from our assumption regarding points of symmetry. But F_{i-1}/F_n lies in between these two fractions. The distance of F_{i-1}/F_n from the point of symmetry, say $1/F_j$, is equal to the distance F_i/F_n from that point of symmetry. Hence this is valid for 55. Similarly it can be made to hold good for 89, ⋯. Hence the theorem.

Theorem 1.2. Whenever we have an interval $[1/F_i, 1/F_{i-1}]$ the denominator of term next to $1/F_i$ is F_{i+2}, and the denominator of the next term is F_{i+4}, then $F_{i+6}\cdots$. We have this until we reach the maximum for that $f\cdot f_n$ sequence, i.e., so long as F_{i+2k} does not exceed F_n. Then the denominator of the term after F_{i+2k} will be the maximum possible term not greater than F_n, but not equal to any of the terms formed, i.e., it's either F_{i+2k+1} or F_{i+2k-1}, say F_j. The denominator of the terms after F_j will be $F_{j-2}, F_{j-4}, \cdots$ till we reach $1/F_{i-1}$. (As an example let us take $[1/3, 1/2]$ in the $f\cdot f$ sequence for 55. Then the denominator of the terms in order are 3, 8, 21, 55, 34, 13, 5, 2).

Proof. The proof of Theorem 1.2 will follow by induction on Theorem 1.1.

Theorem 1.3. (a) If $h/k, h'/k', h''/k''$ are three consecutive fractions of an $f\cdot f$ sequence then

$$\frac{h+h''}{k+k''} = \frac{h'}{k'}$$

if h'/k' is not a point of symmetry.

(b) If h'/k' is a point of symmetry, say $1/F_i$, then

$$\frac{F_{i-2}h + F_{i-1}h''}{F_{i-2}k + F_{i-1}k''} = \frac{h'}{k'}.$$

Proof. Case 1. (From Theorem 1.2) We see that

$$\frac{h}{k} = \frac{F_{i-2}}{F_{j-2}}, \frac{h'}{k'} = \frac{F_i}{F_j}, \frac{h''}{k''} = \frac{F_{i+2}}{F_{j+2}}.$$

In this case

$$\frac{F_{i+2}+F_{i-2}}{F_{j+2}+F_{j-2}} = \frac{{}^*3\cdot F_i}{3\cdot F_j} = \frac{F_i}{F_j} = \frac{h'}{k'}.$$

($^*F_{n+2} + F_{n-2} = 3F_n$ is a property of the Fibonacci sequence. See Hoggatt [1].)

Case 2.

$$\frac{h'}{k'} = \frac{F_i}{F_j}, \frac{h}{k} = \frac{F_{i-2}}{F_{j-2}} \quad \text{and} \quad \frac{h''}{k''} = \frac{F_{i+1}}{F_{j+1}}$$

(from Theorem 1.2). Then

$$\frac{F_{i+1}+F_{i-2}}{F_{j+1}+F_{j-2}} = \frac{2F_i}{2F_j} = \frac{F_i}{F_j} = \frac{h'}{k'}$$

similarly.

Case 3.

$$\frac{h'}{k'} = \frac{F_i}{F_j}, \frac{h}{k} = \frac{F_{i-2}}{F_{j-2}}, \frac{h''}{k''} = \frac{F_{i-1}}{F_{j-1}}$$

(from Theorem 1.2). Therefore

$$\frac{F_{i-1}+F_{i-2}}{F_{j-1}+F_{j-2}} = \frac{F_i}{F_j} = \frac{h'}{k'} .$$

Hence the result.

Proof of 1.3b. Let $h'/k' = 1/F_i$. From Theorem 1.2 it follows that $h''/k'' = 3/F_{i+2}$ and $h/k = 2/F_{i+2}$. Therefore

$$\frac{F_{i-2}h + F_{i-1}h''}{F_{i-2}k + F_{i-1}k''} = \frac{2F_{i-2}+3F_{i-1}}{F_iF_{i+2}} = \frac{F_{i+2}}{F_iF_{i+2}} = \frac{1}{F_i} .$$

Hence the theorem.

Theorem 1.4. If h/k, and h'/k' are two consecutive fractions of an $f\cdot f_n$ sequence then

$$\left|\frac{h-h'}{k-k'}\right| \in f\cdot f_n \qquad (k-k' \neq 0).$$

Proof. Since $f_{(r)n}$ is of the form F_i/F_j, if Theorem 1.4 is to hold, then it is necessary that $|h-h'|$ be equal to F_i and $|k-k'|$ be equal to F_j. Since h/k and h'/k' are members also,

$$h = F_{i_1} , \quad h' = F_{i_2} , \quad k = F_{j_1} , \quad k' = F_{j_2} .$$

Further

$$|F_{j_1} - F_{j_2}| = F_j \qquad \text{and} \qquad |F_{i_1} - F_{i_2}| = F_i .$$

But from the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$ we see that the condition for this is $|i_i - i_2| \leqslant 2$ and $|j_1 - j_2| \leqslant 2$ (but not zero) which follows from Theorem 1.2. Actually

$$\left|\frac{h-h'}{k-k'}\right|$$

are the fractions of the same interval arranged in descending order of magnitude for increasing values of h/k.

Definition 1.4. We now introduce a term "Generating Fraction." If we have a fraction F_i/F_j $(i < j)$. We split F_i/F_j into

$$\frac{F_{i-1}+F_{i-2}}{F_{j-1}+F_{j-2}} .$$

We form from this two fractions F_{i-1}/F_{j-1} and F_{i-2}/F_{j-2} such that F_i/F_j is the mediant of the fractions formed. We continue this process and split the fractions obtained till we reach a state where the numerator is 1. F_i/F_j then amounts to the Generating fraction of the others. We call F_i/F_j as the Generating Fraction of an Interval (G.F.I.) if through this process we are able to get from the G.F.I. all the other fractions of "that" closed interval. We can clearly see a $f\cdot f$ sequence for $F_1, F_2, \cdots, F_n$, F_i/F_n will be a G.F.I. (We also note that F_i/F_j, F_{i-1}/F_{j-1}, F_{i-2}/F_{j-2}, $\cdots$ belong to the same interval because the difference in the suffix of the numerator and denominator is $j-i$). Hence the sequence G.F.I.'s is $F_1/F_n, F_2/F_n, F_3/F_n, \cdots, F_{n-1}/F_n$. We now see some properties concerning G.F.I.'s.

Theorem 1.5. If we form a sequence of the distance between two consecutive G.F.I.'s such a sequence runs thus: $2, 2, 4, 4, 6, 6, 8, 8, \cdots$, i.e., alternate G.F.I.'s are symmetrically placed about a G.F.I.

Theorem 1.6. If we take the first G.F.I., say $f_{(g_1)n}$, then $f_{(g_1+1)n}$ and $f_{(g_1-1)n}$, have the same denominator. For $f_{(g_2)n}$ the second G.F.I. $f_{(g_2+2)n}$, and $f_{(g_2-2)n}$ have the same denominator. In general for $f_{(g_k)n}$ the k^{th} G.F.I. $f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator.

The proofs of theorems 1.5 and 1.6 follow from 1.2.

(NOTE: We can verify that for alternate G.F.I.'s $g_{(g_2)n}, f_{(g_4)n}, f_{(g_6)n}, \cdots, f_{(g_k+k)n}$ and $f_{(g_k-k)n}$ have the same denominator for k is even and the sequence of distance shown above is $2, 2, 4, 4, 6, 6, 8, 8, \cdots$).

PART 2

Definition 2.1. We now define F_{i-2} to be the "factor of the interval"

$$\left[\frac{1}{F_1}, \frac{1}{F_{i-1}}\right] .$$

More precisely the factor of a closed interval is that terms F_z where z is suffix of denominator minus suffix of the numerator, of each fraction of that interval. It can be easily seen (Part 1) that z is a constant.

Lemma 2.1. If $j_1 - i_1 = j_2 - i_2 > 0$, then

$$|F_{j_1}F_{i_2} - F_{j_2}F_{i_2}| = |F_{j_2} - F_{j_1}|\,|F_{j_1} - F_{i_1}| = |F_{j_2} - F_{j_1}|\,|F_{j_2} - F_{i_2}|\ .$$

Proof. We apply Binet's formula that

$$F_n = \frac{a^n - b^n}{a-b}\ ,$$

where

$$a = \frac{1+\sqrt{5}}{2},\qquad b = \frac{1-\sqrt{5}}{2}\ .$$

Then the left-hand side (L.H.S.) of the expression and the right-hand side (R.H.S.) of the expression reduce as follows.

To prove

$$\left|\frac{a^{j_1}-b^{j_1}}{a-b}\cdot\frac{a^{i_2}-b^{i_2}}{a-b} - \frac{a^{j_2}-b^{j_2}}{a-b}\cdot\frac{a^{i_1}-b^{i_1}}{a-b}\right| = \left|\frac{a^{j_2-j_1}-b^{j_2-j_1}}{a-b}\right|\frac{a^{j_1-i_1}-b^{j_1-i_1}}{a-b}$$

because $j_1 - i_1 > 0$, $F_{j_1-i_1}$ is positive and hence can be put within the $|\ |$ sign.

To prove

$$|(a^{j_1}-b^{j_1})(a^{i_2}-b^{i_2}) - (a^{j_2}-b^{j_2})(a^{i_1}-b^{i_1})| = |(a^{j_2-j_1}-b^{j_2-j_1})(a^{j_1-i_1}-b^{j_1-i_1})|$$

the L.H.S. reduces to

$$|a^{j_1+i_2} - a^{j_1}b^{i_2} + b^{j_1+i_2} - b^{j_1}a^{i_2} - a^{j_2+i_1} + a^{j_2}b^{i_1} + b^{j_2}a^{i_1} - b^{j_2+i_1}|$$

$$= |-a^{j_1}b^{i_2} - a^{i_2}b^{j_1} + a^{j_2}b^{i_1} + b^{j_2}a^{i_1}|\ .$$

The R.H.S. reduces to

$$|a^{j_2-i_1} - a^{j_2-j_1}b^{j_1-i_1} + b^{j_2-i_1} - b^{j_2-i_1}a^{j_1-i_1}|\ .$$

This may be simplified further using $ab = -1$ and $j_1 - i_1 = j_2 - i_2$. The R.H.S. is then

$$|a^{j_1}b^{i_2} + b^{j_1}a^{i_2} - a^{j_2}b^{i_1} - b^{j_2}a^{i_1}|\ .$$

We see that L.H.S. = R.H.S. Hence the Lemma.

Corollary. From this we may deduce that if F_{i_1}/F_{j_1} and F_{i_2}/F_{j_2} belong to the same interval, i.e., $j_1 - i_1 = j_2 - i_2$, then

$$F_{j_1}F_{i_2} - F_{j_2}F_{i_1} = F_{|j_2-j_1|}F_{j_2-i_2} = F_{|j_2-j_1|}F_{j_1-i_1}$$

$$F_{i_1}/F_{j_1} < F_{i_2}/F_{j_2}\ .$$

Hence

$$|F_{j_1}F_{i_2} - F_{j_2}F_{i_1}|$$

will be an integral multiple of $F_{j_1-i_1}$ or $F_{j_2-i_2}$ (the factor of that interval) which is the term obtained by the difference in suffixes of the numerator and denominator of each fraction of that interval.

Definition 2.2 We now introduce the term "conjugate fractions." Two fractions h/k and h'/k', h/k and h'/k' are conjugate in an interval

$$\left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right]$$

if the distance of h/k from $1/F_i$ equals the distance of h'/k' from $1/F_{i-1}$ $(h/k \neq h'/k')$.

Corollary. Two consecutive points of symmetry are conjugate with distance zero.

Theorem 2.2. If h/k and h'/k' are conjugate $[1/F_1, 1/F_{i-1}]$ then $kh' - kh' = F_{i-2}$.

Proof. From Part 1, we can easily see that if h/k is of the form

$$\frac{F_{i_1}}{F_{j_1}} \quad \text{then } h'/k' \text{ is } \quad \frac{F_{i_1-1}}{F_{j_1-1}} \quad \cdots \qquad (*)$$

$1/F_i$, and $1/F_{i-1}$ are conjugate. This agrees with (*) since $F_2 = F_1 = 1$. Since the term after $1/F_i$ is F_4/F_{i+2} and the term before $1/F_{i-1}$ is $2/F_{i+1}$, we see it agrees with the statement (*) above. Proceeding in such a fashion we obtain the result (*). Of course we assume here that there exist at least two terms in

$$\left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right] .$$

Hence we can see that any two conjugate to fractions in

$$\left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right]$$

are given by

$$\frac{F_{j-i+2}}{F_j}, \frac{F_{j-i+1}}{F_{j-1}} .$$

We are required to show $|F_jF_{j-1+1} - F_{j-1}F_{j-i+2}| = F_{i-2}$. This will immediately follow from Lemma 2.1.

Theorem 2.3. (a) If h/k and h'/k' are two consecutive fractions in an $f \cdot f_n$ sequence, which belong to $[1/F_i, 1/F_{i-1}]$, then $kh' - hk' = F_{i-2}$.

(b) If h/k and h'/k' are conjugate in an interval $[1/F_i, 1/F_{i-1}]$ $kh' - hk' = F_{i-2}$.

Proof. Theorem 2.3a and 2.3b can be proved using Lemma and Theorem 1.2.

Definition 2.3. If

$$\frac{h}{k} \in \left(\frac{1}{F_i}, \frac{1}{F_{i-1}}\right) ,$$

we define the couplet for h/k as the ordered pair

$$\left[\left(\frac{1}{F_i}, \frac{h}{k}\right), \left(\frac{h}{k}, \frac{1}{F_{i-1}}\right)\right] .$$

Theorem 2.4. In the case of couplets we find that

$$(F_ih) - k = F_pF_{i-2}$$

and

$$k - F_{i-1}h = F_{p+1}F_{i-2} ,$$

where F_p is some Fibonacci number.

Proof. Let h/k be

$$\frac{F_{j-i+2}}{F_j} .$$

Then $(F_ih) - k$ is

(1) $$F_iF_{j-i+2} - F_j = F_pF_{i-2}$$

and let $k - F_{i-1}h$ is

(2) $$F_j - F_{i-1}F_{j-1+2} = F_{p+1}F_{i-2} \cdot$$

Adding (1) and (2) we have

$$F_{i-2}F_{j-1+2} = F_{p+2}F_{i-2} \cdot$$

Therefore $F_{j-i+2} = F_{p+2}$ or $j - i = p$; i.e.,

(3) $$F_iF_{j-i+2} - F_j = F_{j-i}F_{i-2} .$$

We can establish (3) using Lemma 2.1. Hence the proof.

Definition 2.4. We define

$$\left[\left(\frac{1}{F_i}, \frac{h}{k}\right) \left(\frac{h}{k}, \frac{1}{F_{i-1}}\right)\right]$$

and

$$\left[\left(\frac{1}{F_i}, \frac{h'}{k'}\right)\left(\frac{h'}{k'}, \frac{1}{F_{i-1}}\right)\right]$$

to be conjugate couplets if h/k and h'/k' are conjugate fractions of the closed interval

$$\left(\frac{1}{F_i}, \frac{1}{F_{i-1}}\right) .$$

Theorem 2.5. In the case of conjugate couplets if

$$F_ih - k = F_pF_{i-2} \quad \text{and} \quad k - F_{i-1}h = F_{p+1}F_{i-2} ,$$

then

$$F_ih' - k' = F_{p-1}F_{i-2} \quad \text{and} \quad k - F_{i-1}h' = F_pF_{i-2} .$$

Proof. We note that $(j-i)$ in the previous proof is the difference in the suffixes of F_j and F_i. If now

$$h/k = \frac{F_{j-i+2}}{F_j}$$

then $p = j - i$. But since h'/k' is conjugate with h/k,

$$h'/k' = \frac{F_{j-i+1}}{F_{j-1}} .$$

Therefore the constant factor, say F_q in the equation for h'/k', $F_ih' - k = F_qF_{i-2}$ is such that

$$q = j - 1 - i = (j-i) - 1 = p - 1 .$$

Therefore $F_ih' - k' = F_{p-1}F_{i-2}$. Hence $k - F_{i-1}h' = F_pF_{i-2}$ since it follows from Theorem 2.4.

Theorem 2.6. Since we have seen that if h/k and h'/k' are conjugate then the difference in suffixes of their numerators or denominators equals 1, we find

$$\frac{h+h'}{k+k'} \in \left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right] \quad \text{and} \quad \left|\frac{h-h'}{k-k'}\right| \in \left[\frac{1}{F_i}, \frac{1}{F_{i-1}}\right]$$

if

$$h/k, h'/k' \in \left(\frac{1}{F_i}, \frac{1}{F_{i-1}}\right) .$$

Moreover

$$\frac{h+h'}{k+k'}$$

are the fractions of the latter half of the interval arranged in descending order while

$$\left|\frac{h-h'}{k-k'}\right|$$

are the fractions of the first half arranged in ascending order, for increasing values of h/k.

PART 3

We now give a generalized result concerning "sequence of distances."

Theorem 3.1a. Points of symmetry if they are of the form $f_{(r)n}$ then

$$r \in \left\{2, 3, 5, 8, 12, 17, \cdots\right\} .$$

Or the sequence of distance between two consecutive points of symmetry will be

$$1, 2, 3, 4, 5, 6, \cdots ,$$

an Arithmetic progression with common difference 1.

Theorem 3.1b. The sequence of distance for fractions with common numerator F_{2n-1} or F_{2n} is

$$2n-1, 2n, 2n+1, \cdots .$$

Proof. To prove Theorem 3.1a we have to show that if there are n terms in an interval then there are $(n+1)$ terms in the next.

Let there be p terms of the form F_i/F_j. It is evident that there are $p+1$ terms of the form F_{i+1}/F_j. But these $(p+1)$ terms of the form F_{i+1}/F_j are in an interval next to that in which the p terms of the form F_i/F_j lie. So the sequence is an AP with common difference 1. Moreover, the second term is always $1/F_n$ (evident). Hence the result. (Note: $j-i$ is assumed constant.)

If we fix the numerator to be 2 and take the sequence

$$\frac{2}{F_n}, \frac{2}{F_{n-1}}, \frac{2}{F_{n-2}}, \cdots, \frac{2}{3}$$

then the sequence of distance between two consecutive such fractions is $3, 4, 5, \cdots$.

From Theorem 1.2 (Part 1) it follows that $2/F_i$ lies just before a point of symmetry, say $1/F_j$. Since we have seen the sequence of distances concerning points of symmetry it will follow that here too the common difference is 1. The first term is 3 for there are two terms between $2/F_n$ and $2/F_{n-1}$. The inequality

$$\frac{2}{F_n} < \frac{1}{F_{n-2}} < \frac{3}{F_n} < \frac{2}{F_{n-1}}$$

can be established. Hence the result.

In a similar fashion we find that the sequence of distance for numerator 3 is $3, 4, 5, \cdots$.

We shall give a table and the generalization

Numerator	Sequence of Distance
F_1 or F_2	$1, 2, 3, 4, 5, \cdots$
F_3 or F_4	$3, 4, 5, 6, \cdots$
F_5 or F_6	$5, 6, 7, 8, \cdots$
F_{2n-1} or F_{2n}	$2n-1, 2n, 2n+1, 2n+2, \cdots$.

Definition 3.1. Just as we defined an interval, we now define a "stage" as the set of $f \cdot f$ fractions lying between two consecutive G.F.I.'s. The stage may be closed or open depending upon the inclusion or omission of the G.F.I.'s.

Since the sequence of distance of G.F.I.'s is $2, 2, 4, 4, 6, 6, \cdots$, it is possible for two consecutive "stages" to have equal numbers of terms. We define two stages:

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right] \qquad \text{and} \qquad \left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

to be conjugate stages if the distance of F_i/F_n from F_{i-1}/F_n equals the distance of F_{i+1}/F_n from F_i/F_n. That is the number of terms in two conjugate stages are equal. We call a stage comparison of both these stages as a "complex stage." Let us now investigate properties concerning stages. If we have complex stage

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

then we define two fractions h/k and h'/k' to be "corresponding" if

$$\frac{h}{k} \in \left(\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right)$$

and

$$\frac{h'}{k'} \in \left(\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right)$$

and if the distance of h/k from F_{i-1}/F_n is equal to the distance of h'/k' from F_i/F_n.

Theorem 3.2. Two corresponding fractions have the same numerator. If h/k and h'/k' are corresponding fractions then $h = h'$.

Proof. This will follow from 1.2 (part 1).

Let F_{i-1}/F_n be the maximum reached in its interval so that F_{i-1}/F_{n-1} will be the maximum for the interval in which F_i/F_n belongs. (where by maximum we mean the term with denominator F_{i+2k} in the sense of Theorem 1.2). The term next to F_{i-1}/F_n is F_{i-2}/F_{n-1}. Similarly the term next to F_i/F_n is F_{i-2}/F_{n-2}. But these fractions are corresponding in such a fashion that we obtain the result.

Now F_{i-1}/F_n has necessarily to be the maximum in its interval. Since we have considered conjugate stages i is odd. Using Theorem 1.2 it can be established that alternate G.F.I.'s are maximum in their interval and that too, when suffix of numerator is even $(i-1$ is even).

Definition 3.2. Since the number of terms in a stage is odd, we define h/k to be the middle point of a stage

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right]$$

if it is equidistant from both G.F.I.'s. We can deduce from this that h/k is a point of symmetry since F_{i-1}/F_n, and F_i/F_n have the same denominator. So the middle point of a stage is a point of symmetry.

Corollary. If two conjugate stages are taken then their middle points are corresponding. (This follows from the definition). But their numerators should be equal. This is so, for the middle points are points of symmetry whose numerator is 1. This agrees with the result proved.

Definition 3.3. Two fractions h/k and h'/k' are conjugate in a complex stage if the distance of h/k from F_{i-1}/F_n equals the distance of h'/k' from F_{i+1}/F_n, $h/k < h'/k'$ and the complex stage being

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

Taking their middle points

$$\left[\frac{1}{F_p}, \frac{1}{F_{p+1}}\right]$$

we can see that fractions conjugate in this interval are conjugate in the complex stage. Further we saw that for conjugate fractions of the interval, h/k, h'/k',

$$\frac{h+h'}{k+k'}$$

re fractions of the latter half of the interval arranged in descending order, and

$$\left|\frac{h-h'}{k-k'}\right|$$

are fractions of the first half arranged in ascending order for increasing values of h/k.

Theorem 3.3. For conjugate fractions h/k and h'/k' lying in the outer half of the stage we see that

$$\frac{h+h'}{k+k'}$$

are fractions of the latter half of the interval in ascending order while

$$\left|\frac{h-h'}{k-k'}\right|$$

are fractions of the first half in descending order for increasing values of h/k. We here only give a proof to show that

$$\frac{h+h'}{k+k'} \quad \text{and} \quad \frac{h-h'}{k-k'}$$

are in the interval but do not prove the order of arrangement.

Proof. For h/k, h'/k', in the inner half the proof has been given (previous part). The middle point of

$$\left[\frac{F_{i-1}}{F_n}, \frac{F_i}{F_n}\right]$$

is $1/F_{n-i+2}$. Similarly the middle point of

$$\left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right]$$

is $1/F_{n-i+2}$. That two conjugate fractions of the outer half of a conjugate stage differ in suffix by 1 can be established. That is, to say, if

$$\frac{h}{k} = \frac{F_{j-(n-i)-1}}{F_j}$$

then

$$\frac{h'}{k'} = \frac{F_{j-(n-i)}}{F_{j-1}} \qquad \frac{h+h'}{k+k'} = \frac{F_{j-(n-i)+1}}{F_{j+1}} \in I,$$

where I is the interval $[1/F_p,\ 1/F_{p+1}]$ and

$$\frac{h-h'}{k-k'} = \frac{F_{j-(n-i)-2}}{F_{j-2}} \in I.$$

Hence the proof.

Definition 3.4. In an $f \cdot f$ sequence of order F_n, $[F_i/F_n,\ F_{i+1}/F_n]$ represents a stage. Let us take an $f \cdot f$ sequence of order F_{n+1}. If there we take a stage $[F_i/F_{n+1},\ F_{i+1}/F_{n+1}]$, then we say the two stages are corresponding stages. More generally in an $f \cdot f$ sequence of order F_n and an $f \cdot f$ sequence of order F_{n+k},

$$\left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right], \quad \left[\frac{F_i}{F_{n+k}}, \frac{F_{i+1}}{F_{n+k}}\right]$$

are corresponding stages. We stage here properties of corresponding stages. These can be proved using Theorem 1.2.

Theorem 3.4a. If

$$\left[\frac{F_i}{F_n}, \frac{F_{i+1}}{F_n}\right] \quad \text{and} \quad \left[\frac{F_i}{F_{n+k}}, \frac{F_{i+1}}{F_{n+k}}\right]$$

are corresponding stages then the number of terms in both are equal.

Theorem 3.4b. There exists a one-one correspondence between the denominators of these stages. If the denominator of the q^{th} term of $[F_i/F_n,\ F_{i+1}/F_n]$ is F_j then the denominator of the q^{th} term of

$$\left[\frac{F_i}{F_{n+k}}, \frac{F_{i+1}}{F_{n+k}}\right]$$

is F_{j+k}.

We can extend this idea further and produce a one-one correspondence between

$$\left[\frac{F_i}{F_n}, \frac{F_{i+m}}{F_n}\right] \quad \text{and} \quad \left[\frac{F_i}{F_{n+k}}, \frac{F_{i+m}}{F_{n+k}}\right], \quad \text{where} \quad \left[\frac{a}{b}, \frac{c}{d}\right]$$

stands for the set of fractions between a/b and c/d inclusive of both. A further extension would give that given two $f \cdot f$ sequences, one of order F_n, and the other of order F_{n+k}.

Theorem 3.5a. The numerator of the r^{th} term of the first sequence equals the numerator of the r^{th} term of the second.

Theorem 3.5b. If the denominator of the r^{th} term of the first sequence is F_j, *then the denominator of the* r^{th} term of the second series is F_{j+1}. Precisely

(a) the numerator of $f_{(r)n}$ is equal to the numerator of $f_{(r)n+k}$

(b) if the denominator of $f_{(r)n} = F_j$, the denominator of $f_{(r)n+k} = F_{j+k}$.

This can be proved using 1.2. We can arrive at the same result by defining corresponding intervals.

Definition 3.5. Two intervals, $[1/F_i,\ 1/F_{i+1}]$ in an $f \cdot f$ sequence of order F_n and $[1/F_{i+k},\ 1/F_{i+k}]$ in an $f \cdot f$ sequence of order F_{n+k} are defined to be corresponding intervals.

The same one-one correspondence as in the case of corresponding stages exists for corresponding intervals. We can extend this correspondence in a similar manner to the entire $f \cdot f$ sequence and prove that

(a) the numerator of $f_{(r)n}$ is equal to the numerator of $f_{(r)n+k}$,

(b) if the denominator of $f_{(r)n} = F_j$, the denominator of $f_{(r)n+k} = F_{j+k}$.

(c) GENERALIZED $f \cdot f$ SEQUENCE. We defined the $f \cdot f$ sequence in the interval $[0,1]$. We now define it in the interval $[0,\infty]$.

Definition 3.6. The $f \cdot f$ sequence of order F_n is the set of all functions F_i/F_j, $j \leqslant n$ arranged in ascending order of magnitude $i,j \geqslant 0$. If $i < j$ then the $f \cdot f$ sequence is in the interval $[0,1]$. The basic properties of the $f \cdot f$ sequence for $[0,1]$ are retained with suitable alterations

Theorem 3.6.1. $f_{(r)n}$ is a point of symmetry if $f_{(r+1)n}$ and $f_{(r-1)n}$ have the same numerator (beyond $1/1$). If $f_{(r)n}$ is a point of symmetry then $f_{(r+k)n}$ and $f_{(r-k)n}$ have the same numerator, if each fraction does not pass beyond the next G.F.I. in either side (beyond $1/1$).

Theorem 3.6.2. A G.F.I. is a fraction with denominator F_n .

Theorem 3.6.3. A point of symmetry has either numerator or denominator 1.

Theorem 3.6.4. Beyond $1/1$, any interval is given by $[F_{n-1}/1, F_n/1]$. The factor of this interval is again F_{n-2}.

Theorem 3.6.5. The two basic properties

(a)
$$\frac{h+h''}{k+k''} = \frac{h'}{k'}$$

and

(b)
$$kh - hk' = F_{n-2}$$

are retained.

Theorem 3.6.6. If (a) is not good for h'/k' being a point of symmetry then

$$\frac{h'}{k'} = \frac{F_{n-1}h'' + F_{n-2}h}{F_{n-1}k'' + F_{n-2}k} \qquad \text{if} \qquad \frac{h}{k} < \frac{h'}{k'} < \frac{h''}{k''}; \ \frac{h'}{k'} = \frac{F_n}{1} \quad .$$

For a pertinent article by this author entitled "Approximation of Irrationals using Farey Fibonacci Fractions," see later issues.

f·f Sequence of Order 5

$$\frac{0}{3}, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{1}{1}$$

f·f Sequence of Order 8

$$\frac{0}{5}, \frac{1}{8}, \frac{1}{5}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$$

f·f Sequence of Order 13

$$\frac{0}{8}, \frac{1}{13}, \frac{1}{8}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1}$$

f·f Sequence of Order 21

$$\frac{0}{13}, \frac{1}{21}, \frac{1}{13}, \frac{2}{21}, \frac{1}{8}, \frac{3}{21}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13}, \frac{5}{21}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1} .$$

f·f Sequence of Order 34

$$\frac{0}{21}, \frac{1}{34}, \frac{1}{21}, \frac{2}{34}, \frac{1}{13}, \frac{3}{34}, \frac{2}{21}, \frac{1}{8}, \frac{3}{21}, \frac{5}{34}, \frac{2}{13}, \frac{1}{5}, \frac{3}{13},$$

$$\frac{8}{34}, \frac{5}{21}, \frac{2}{8}, \frac{1}{3}, \frac{3}{8}, \frac{8}{21}, \frac{13}{34}, \frac{5}{13}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{13}{21}, \frac{5}{8}, \frac{2}{3}, \frac{1}{1} .$$

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