CHEBYSHEV POLYNOMIALS AND RELATED SEQUENCES

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1. A COMBINATORIAL APPROACH

In [3], the nonzero coefficients of the Chebyshev polynomials $T_n(x) = \cos n\theta$, $\cos\theta = x$, which satisfy the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ since $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$, are arranged in left-adjusted triangular form. The first seven rows of the array are

$n \backslash k$	0	1	2	3
0	1			
1	1			
2	2	−1		
3	4	−3		
4	8	−8	1	
5	16	−20	5	
6	32	−48	18	−1

Furthermore, letting $a_{n,k}$ be the element in the n^{th} row and k^{th} column, it is shown in [3] that

$$a_{n,k} = (-1)^k \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k-1} \tag{1.1}$$

and

$$a_{n,k} = 2a_{n-1,k} - a_{n-2,k-1}\ . \tag{1.2}$$

In this section, we discuss several linear recurrences which arise as a result of a careful examination of the triangular array. The validity of these linear recurrences is established by means of common combinatorial identities.

Summing along the rising diagonals, we obtain the sequence $1, 1, 2, 3, 5, 8, 13, \cdots$, which appears to be the sequence of Fibonacci numbers. To show that this is in fact the case, we first observe that the sum of the n^{th} rising diagonal is given by

$$f_n = \begin{cases} 1, & n = 1 \text{ or } 2 \\ \displaystyle\sum_{k=0}^{M} a_{n-k-1,k}, & M = \left[\dfrac{n-1}{3}\right],\ n \geqslant 3\ . \end{cases} \tag{1.3}$$

We now verify that $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$.

In [2], we find the following combinatorial identities

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{1.4}$$

and

(1.5) $$\binom{n-k}{k} + \binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k} .$$

Using (1.1) together with (1.3) and applying (1.5) and then (1.4) twice, we have,

$$f_n = \sum_{k=0}^{M} (-1)^k \frac{n-k-1}{n-2k-1}\binom{n-2k-1}{k} 2^{n-3k-2}$$

$$= \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-2}{k} + 2\binom{n-2k-2}{k-1}\right] 2^{n-3k-2}$$

$$= \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-2}{k} + \binom{n-2k-3}{k-1}\right] 2^{n-3k-3}$$

(1.6) $$+\sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-3}{k} + 4\binom{n-2k-2}{k-1}\right] 2^{n-3k-3}$$

$$= f_{n-1} + \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-3}{k} + \binom{n-2k-4}{k-1}\right] 2^{n-3k-4}$$

$$+\sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-4}{k} + 8\binom{n-2k-2}{k-1}\right] 2^{n-3k-4}$$

$$= f_{n-1} + f_{n-2} + \sum_{k=0}^{M} (-1)^k \left[\binom{n-2k-4}{k} + 8\binom{n-2k-2}{k-1}\right] 2^{n-3k-4} .$$

Since the first and last terms cancel for successive integral values in the last sum, and because

$$n-4 < n-1 \leqslant 3M \quad \text{implies that} \quad n-2M-4 < M,$$

the last sum has value zero so that

(1.7) $$f_n = f_{n-1} + f_{n-2}, \qquad n \geqslant 3 .$$

The sequence of the sums of the rising diagonals in absolute value, denoted by $\{u_n\}_{n=1}^{\infty}$, is $1,1,2,5,11,24,53,\cdots$ and it appears to satisfy the recurrence relation

(1.8) $$u_1 = u_2 = 1, \quad u_3 = 2, \quad 2u_{n-1} + u_{n-3} = u_n, \quad n \geqslant 4 .$$

By the definition of u_n, (1.1), and (1.3), we see for $n \geqslant 4$, following an argument similar to that of (1.6), that,

$$u_n = \sum_{k=0}^{M} \frac{n-k-1}{n-2k-1}\binom{n-2k-1}{k} 2^{n-3k-2} = \sum_{k=0}^{M} \left[\binom{n-2k-2}{k} + 2\binom{n-2k-2}{k-1}\right] 2^{n-3k-2}$$

(1.9) $$= 2\sum_{k=0}^{M} \left[\binom{n-2k-2}{k} + \binom{n-2k-3}{k-1}\right] 2^{n-3k-3} + \sum_{k=0}^{M}\left[2\binom{n-2k-2}{k-1} - \binom{n-2k-3}{k-1}\right] 2^{n-3k-2}$$

$$= 2u_{n-1} + \sum_{k=0}^{M-1}\left[2\binom{n-2k-4}{k} - \binom{n-2k-5}{k}\right] 2^{n-3k-5} = 2u_{n-1} + \sum_{k=0}^{M}\left[\binom{n-2k-4}{k}\right.$$

$$\left. + \binom{n-2k-5}{k-1}\right] 2^{n-3k-5} = 2u_{n-1} + u_{n-3}$$

and (1.8) is proved.

Let w_n be the sum of the terms along the n^{th} falling diagonal. The terms of $\{w_n\}_{n=1}^{\infty}$ appear to be given by

$$w_n = \begin{cases} 1, & n = 1 \\ 0, & n \geq 2 \end{cases} . \tag{1.10}$$

To show that $w_n = 0$ for $n \geq 2$, we observe that

$$\begin{aligned} w_n &= \sum_{k=0}^{n-1} a_{n+k-1,k} = \sum_{k=0}^{n-1} (-1)^k \left[\binom{n-1}{k} + \binom{n-2}{k-1} \right] 2^{n-k-2} \\ &= \tfrac{1}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} 2^{n-k-1} - \tfrac{1}{2} \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} 2^{n-k-2} \\ &= \frac{(2-1)^{n-1}}{2} - \frac{(2-1)^{n-2}}{2} = 0 \end{aligned} \tag{1.11}$$

and (1.10) is proved.

Letting q_n be the sum of the absolute value of the terms along the n^{th} falling diagonal, we see that the terms of $\{q_n\}_{n=1}^{\infty}$ are $1, 2, 6, 18, 54, 162, 486, \cdots$ and it appears as if we have

$$q_n = \begin{cases} 1, & n = 1 \\ 2 \cdot 3^{n-2}, & n \geq 2 \end{cases} . \tag{1.12}$$

By the definition of q_n and (1.11), we have

$$\begin{aligned} q_n &= \sum_{k=0}^{n-1} |a_{n+k-1,k}| = \tfrac{1}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} 2^{n-k-1} + \tfrac{1}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} 2^{n-k-2} \\ &= \frac{(2+1)^{n-1}}{2} + \frac{(2+1)^{n-2}}{2} = 2 \cdot 3^{n-2} \end{aligned} \tag{1.13}$$

so that (1.12) is true.

It is easy to determine the row sum r_n because, as is pointed out in [3], the sums are all one since $\cos n0 = 1$. The last sequence of this section, denoted by $\{p_n\}_{n=1}^{\infty}$, deals with the sums of the absolute values of the terms of the rows, and the first few terms of the sequence are $1, 1, 3, 7, 17, 41, 91, \cdots$. It appears as if we have

$$p_1 = p_2 = 1, \quad p_n = 2p_{n-1} + p_{n-2}, \quad n \geq 3, \tag{1.14}$$

which is a generalized Pell sequence where the Pell numbers P_n are given by the recurrence relation

$$P_1 = 1, \quad P_2 = 2, \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3. \tag{1.15}$$

The first few terms of the sequence are $1, 2, 5, 12, 29, 70, 169, \cdots$. Letting $P_{-1} = 1$ and $P_0 = 0$, it is easy to establish by mathematical induction that

$$p_n = P_{n-1} + P_{n-2} = P_n - P_{n-1} \tag{1.16}$$

and

$$P_n = \sum_{i=1}^{n} p_n . \tag{1.17}$$

To verify (1.14), we use (1.2) and observe that

$$|a_{n,k}| = 2|a_{n-1,k}| + |a_{n-2,k-1}| \tag{1.18}$$

so that with $N = [n/2]$, we have

$$p_n = \sum_{k=0}^{N} |a_{n,k}| = 2 \sum_{k=0}^{N} |a_{n-1,k}| + \sum_{k=0}^{N} |a_{n-2,k-1}| = 2p_{n-1} + \sum_{k=0}^{N-1} |a_{n-2,k}| . \tag{1.19}$$

However, $|a_{n-2,N}| = 0$ because $n - 2 < n \leq 2N$ implies that $n - 2 - N < N$. Hence,

(1.20)
$$p_n = 2p_{n-1} + p_{n-2}\,.$$

2. GENERATING FUNCTIONS

In a personal correspondence, V.E. Hoggatt, Jr., pointed out that the relationships of Section 1 could be established by means of generating functions.

Let $G_k(x)$ be the generating function for the k^{th} column. Following standard techniques, it is easy to show that

(2.1)
$$G_0(x) = \frac{1-x}{1-2x}$$

and, with the aid of (1.2) that

(2.2)
$$G_k(x) = \frac{-G_{k-1}(x)}{1-2x}\,.$$

Employing mathematical induction together with (2.1) and (2.2), we have

(2.3)
$$G_k(x) = \left(\frac{-1}{1-2x}\right)^k\left(\frac{1-x}{1-2x}\right), \quad k \geqslant 0.$$

Adding along the rising diagonals is equivalent to

(2.4)
$$\begin{aligned}\sum_{k=0}^{\infty} x^{3k}G_k(x) &= \sum_{k=0}^{\infty}\left(\frac{1-x}{1-2x}\right)\left(\frac{-x^3}{1-2x}\right)^k \\ &= \left(\frac{1-x}{1-2x}\right) \div \left(1+\frac{x^3}{1-2x}\right) \\ &= (1-x-x^2)^{-1}\,.\end{aligned}$$

Since
$$(1-x-x^2)^{-1}$$
is the generating function for the Fibonacci sequence, we have an alternate proof of (1.7).

Letting

(2.5)
$$G_k^*(x) = \left(\frac{1-x}{1-2x}\right)\left(\frac{1}{1-2x}\right)^k,$$

we see that adding along rising diagonals with all signs positive is equivalent to

(2.6)
$$\sum_{k=0}^{\infty} x^{3k}G_k^*(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1-\frac{x^3}{1-2x}\right) = \frac{1-x}{1-2x-x^3}$$

which verifies (1.8) since $(1-x)(1-2x-x^3)^{-1}$ is the generating function for $\{u_n\}_{n=1}^{\infty}$.

To verify (1.10) and (1.12), we recognize that

(2.7)
$$\sum_{k=0}^{\infty} x^k G_k(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1+\frac{x}{1-2x}\right) = 1\,,$$

where 1 is the generating function for $\{w_n\}_{n=1}^{\infty}$ while

(2.8)
$$\sum_{k=0}^{\infty} x^k G_k^*(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1-\frac{x}{1-2x}\right) = \frac{1-x}{1-3x}\,,$$

where $(1-x)(1-3x)^{-1}$ is the generating function for $\{q_n\}_{n=1}^{\infty}$.

Since

(2.9)
$$\sum_{k=0}^{\infty} x^{2k}G_k(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1+\frac{x^2}{1-2x}\right) = (1-x)^{-1}$$

we have an alternate proof that the row sums are all one. Furthermore,

(2.10)
$$\sum_{k=0}^{\infty} x^{2k} G_k^*(x) = \left(\frac{1-x}{1-2x}\right) \div \left(1 - \frac{x^2}{1-2x}\right) = \frac{1-x}{1-2x-x^2},$$

where $(1-x)(1-2x-x^2)^{-1}$ is the generating function for $\{p_n\}_{n=1}^{\infty}$. Hence, we have an alternate proof of (1.14).

In conclusion, we note that

(2.11)
$$\sum_{n=0}^{\infty} P_{n-1}x^n + \sum_{n=0}^{\infty} P_n x^n = \frac{1-2x}{1-2x-x^2} + \frac{x}{1-2x-x^2} = \frac{1-x}{1-2x-x^2} = \sum_{n=0}^{\infty} p_{n+1}x^n$$

and we have a generating function proof of (1.16).

3. ANOTHER ARRAY

If we let
$$Q_n(x) = \frac{\sin n\theta}{\sin\theta}, \qquad x = \cos\theta,$$
and use
$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta \sin n\theta,$$
we see that
$$Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x)$$
and $Q_n(x)$ is a polynomial in x.

The first eight rows of the nonzero coefficients of the polynomials $Q_n(x)$ in left-adjusted triangular form are

$n \backslash k$	0	1	2	3
1	1			
2	2			
3	4	−1		
4	8	−4		
5	16	−12	1	
6	32	−32	6	
7	64	−80	24	−1
8	128	−192	80	−8

Letting $b_{n,k}$ be the element in the n^{th} row and k^{th} column, it can be shown, as in [3], that

(3.1)
$$b_{n,k} = 2b_{n-1,k} - b_{n-2,k-1}$$
and

(3.2)
$$b_{n,k} = (-1)^k \binom{n-k-1}{k} 2^{n-2k-1}.$$

The six linear recurrences of Section 1, relative to the $Q_n(x)$ array, are

(3.3) $$F_1 = 1, \quad F_2 = 2, \quad F_n = F_{n-1} + F_{n-2} + 1, \quad n \geq 3$$

(3.4) $$U_1 = 1, \quad U_2 = 2, \quad U_3 = 4, \quad U_n = 2U_{n-1} + U_{n-3}, \quad n \geq 4$$

(3.5) $$W_n = 1, \quad n \geq 1$$

(3.6) $$Q_n = 3^{n-1}, \quad n \geq 1$$

(3.7) $$R_n = n, \quad n \geq 1,$$

and

(3.8) $$P_1 = 1, \quad P_2 = 2, \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3$$

which is the sequence of Pell numbers given in (1.15).

The preceding six linear recurrences can be verified by using combinatorial arguments like those of Section 1 or by means of generating functions as in Section 2 where the column generators of the $Q_n(x)$ table are given by

(3.9) $$H_k(x) = \frac{1}{1-2x}\left(\frac{-1}{1-2x}\right)^k, \quad k \geq 0$$

and

$$H_k^*(x) = \frac{1}{1-2x}\left(\frac{1}{1-2x}\right)^k, \quad k \geqslant 0 \tag{3.10}$$

if we want all positive values. Hence, the details are omitted.

4. CONCLUDING REMARKS

Equations (1.16) and (1.17) relate the sequences of (1.14) and (3.8). Similar relationships, which can be proved by mathematical induction, also hold for the other five recurrences. That is,

$$f_n = F_n - F_{n-1} \quad \text{and} \quad F_n = \sum_{i=1}^{n} f_i \tag{4.1}$$

$$u_n = U_n - U_{n-1} \quad \text{and} \quad U_n = \sum_{i=1}^{n} u_i \tag{4.2}$$

$$w_n = W_n - W_{n-1} \quad \text{and} \quad W_n = \sum_{i=1}^{n} w_i \tag{4.3}$$

$$q_n = Q_n - Q_{n-1} \quad \text{and} \quad Q_n = \sum_{i=1}^{n} q_i \tag{4.4}$$

$$r_n = R_n - R_{n-1} \quad \text{and} \quad R_n = \sum_{i=1}^{n} r_i \cdot \tag{4.5}$$

Since Eq. (3.9) is $(1-x)^{-1}$ times Eq. (2.3), it can be shown that the entries in the $Q_n(x)$ table are partial sums of the column entries of the $T_n(x)$ table. Hence,

$$b_{n+2k,k} = \sum_{j=0}^{n-1} a_{j+2k,k} \tag{4.6}$$

which gives rise to the combinatorial identity

$$2^n \binom{n+k}{k} = \sum_{j=0}^{n} \left(\frac{j+2k}{j+k}\right)\binom{j+k}{k} 2^{j-1} . \tag{4.7}$$

An interesting consequence of (4.6) since the $b_{n,k}$ and $a_{n,k}$ are respectively the coefficients of the polynomials $Q_n(x)$ and $T_n(x)$ is the identity

$$\sum_{j=0}^{n} \cos^{n-j}\theta \cos j\theta = \frac{\sin (n+1)\theta}{\sin \theta} . \tag{4.8}$$

REFERENCES

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