

SYMMETRIC SEQUENCES

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This paper deals with integer sequences governed by linear recursion relations. To avoid useless duplication, sequences with terms having a common factor greater than one will be considered equivalent to the sequence with the greatest common factor of the terms eliminated. The recursion relation governing a sequence will be taken as the recursion relation of lowest order which it obeys.

Symmetric sequences are of two types:

A. Sequences with an Unmatched Zero Term

$$(1) \quad \dots T_{-3}, T_{-2}, T_{-1}, T_0, T_1, T_2, T_3, \dots$$

with

$$T_n = T_{-n}$$

B. Sequences with All Matched Terms

$$(2) \quad \dots T_{-3}, T_{-2}, T_{-1}, T_1, T_2, T_3, \dots$$

FIRST-ORDER SEQUENCES

The recursion relation of the first order is:

$$(3) \quad T_{n+1} = aT_n$$

which will have all terms integers only if $a = \pm 1$. The only sequences governed by such relations subject to the initial restrictions given above are:

$$\dots 1, 1, 1, 1, 1, 1, \dots$$

$$\dots -1, 1, -1, 1, -1, 1, \dots$$

These sequences and the sequence $\dots 0, 0, 0, 0, \dots$ will be eliminated from consideration in the work that follows.

SECOND-ORDER SEQUENCES

For a recursion relation

$$T_{n+1} = aT_n + bT_{n-1}$$

to have all integer terms, the quantity b must be $+1$ or -1 . The same applies to sequences of higher order. These will be denoted Case I ($+1$) and Case II (-1).

Case I.

$$T_{n+1} = aT_n + T_{n-1}$$

A. Zero Term

$$T_0 = T_2 - aT_1, \quad T_{-1} = T_1 - aT_0 = T_1 - aT_2 + a^2T_1 = T_1, \quad a(aT_1 - T_2) = 0.$$

Thus either $a = 0$ or $T_0 = 0$. $a = 0$ leads to sequences such as:

$$\dots 2, 3, 2, 3, 2, 3, 2, 3, \dots$$

If $T_0 = 0$,

$$T_{-2} = T_2 = T_0 - aT_{-1} = -aT_1.$$

Hence $T_2 = aT_1$ and $T_2 = -aT_1$ with the result that $a = 0$.

B. No Zero Term

$$T_{-1} = T_2 - aT_1 = T_1, \quad (a+1)T_1 = T_2, \quad T_{-2} = T_2 = T_1 - aT_{-1} = (1-a)T_1.$$

Therefore $aT_1 = 0$. If $T_1 = 0$, all the terms are zero. If $a = 0$, we have the type of sequence given above for this value.

Case II.

$$T_{n+1} = aT_n - T_{n-1}.$$

A. Zero Term

$$(4) \quad \begin{aligned} T_0 &= aT_1 - T_2, & T_{-1} &= T_1 = aT_0 - T_1 = a^2T_1 - aT_2 - T_1 \\ (a^2 - 2)T_1 - aT_2 &= 0, & T_{-2} &= T_2 = aT_{-1} - T_0 = aT_{-1} - aT_1 + T_2 = T_2. \end{aligned}$$

If symmetry holds up to T_n , then

$$T_{-n-1} = aT_{-n} - T_{-n+1} = aT_n - T_{n-1} = T_{n+1}$$

and hence the entire sequence will be symmetrical.

EXAMPLES

For any value of a , select T_1 and T_2 to satisfy (4) in order to generate a symmetric sequence. Thus for $a = 3$, $7T_1 = 3T_2$, giving the sequence:

$$\dots 47, 18, 7, 3, 2, 3, 7, 18, 47, \dots$$

governed by

$$T_{n+1} = 3T_n - T_{n-1}.$$

For $a = 8$, $62T_1 = 8T_2$, giving the sequence:

$$\dots 1921, 244, 31, 4, 1, 4, 31, 244, 1921, \dots$$

governed by $T_{n+1} = 8T_n - T_{n-1}$.

B. No Zero Term

The relations

$$T_{-1} = T_1 = aT_1 - T_2 \quad \text{and} \quad T_{-2} = aT_{-1} - T_1$$

both lead to

$$(a-1)T_1 = T_2.$$

If $T_{-n} = T_n$ holds up to n , then

$$T_{-n-1} = aT_{-n} - T_{-n+1} = aT_n - T_{n-1} = T_{n+1}$$

and the symmetry will be maintained throughout the sequence.

For $a = 5$, $T_2 = 4T_1$ giving a sequence

$$\dots 19, 4, 1, 1, 4, 19, 91, 436, \dots$$

governed by

$$T_{n+1} = 5T_n - T_{n-1}.$$

THIRD-ORDER SEQUENCES

Case I.

$$T_{n+1} = aT_n + bT_{n-1} + T_{n-2}.$$

A. Zero Term

$$\begin{aligned} T_{n-2} &= T_{n+1} - aT_n - bT_{n-1}, & T_0 &= T_3 - aT_2 - bT_1, \\ T_{-1} &= T_1 = T_2 - aT_1 - bT_0 = T_2 - aT_1 - bT_3 + abT_2 + b^2T_1 \\ & & (b^2 - a - 1)T_1 + (ab + 1)T_2 &= bT_3. \end{aligned}$$

(5)

Also

$$T_{-2} = T_2 = T_1 - aT_0 - bT_{-1} = T_1 - aT_3 + a^2T_2 + abT_1 - bT_1$$

from which

(6)

$$(ab - b + 1)T_1 + (a^2 - 1)T_2 = aT_3$$

$$T_{-3} = T_3 = T_0 - aT_{-1} - bT_{-2} = T_3 - aT_2 - bT_1 - aT_1 - bT_2$$

so that

$$(7) \quad (a+b)(T_1+T_2) = 0.$$

Equation (7) will hold if $b = -a$ which makes (5) and (6):

$$(5') \quad (a^2 - a - 1)T_1 + (1 - a^2)T_2 = -aT_3$$

$$(6') \quad (-a^2 + a + 1)T_1 + (a^2 - 1)T_2 = aT_3$$

which are the same relation. Since

$$T_4 = aT_3 - bT_2 + T_1 \quad \text{and} \quad T_{-4} = T_{-1} - aT_{-2} - bT_{-3} = T_1 - aT_2 + aT_3 = T_4$$

the symmetry persists up to this point. An entirely similar argument shows that it holds in general.

EXAMPLE. For a given value of a , many symmetric sequences can be determined. For $a = 5$,

$$19T_1 - 24T_2 = -5T_3$$

from which one may derive any number of symmetric sequences obeying the relation

$$T_{n+1} = 5T_n - 5T_{n-1} + T_{n-2}.$$

Examples are:

$$\dots 1350, 361, 96, 25, 6, 1, 0, 1, 6, 25, 96, 361, 1350, \dots$$

$$\dots 363, 98, 27, 8, 3, 2, 3, 8, 27, 98, 363, \dots, \quad \dots 362, 97, 26, 7, 2, 1, 2, 7, 26, 97, 362, \dots$$

B. No Zero Term

$$(8) \quad T_{n+1} = aT_n + bT_{n-1} + T_{n-2}, \quad T_{n-2} = T_{n+1} - aT_n - bT_{n-1}, \quad T_{-1} = T_1 = T_3 - aT_2 - bT_1$$

$$(b+1)T_1 + aT_2 = T_3$$

$$(9) \quad T_{-2} = T_2 = T_2 - aT_1 - bT_{-1}$$

$$(a+b)T_1 = 0$$

which is satisfied if $b = -a$

$$(10) \quad T_{-3} = T_3 = T_1 - aT_{-1} - bT_{-2}$$

$$T_3 = (1-a)T_1 + aT_2$$

which agrees with (8) when $b = -a$.

If the symmetry holds to $T_n = T_{-n}$, then

$$T_{-n-1} = T_{-n+2} - aT_{-n+1} + aT_{-n} = T_{n-2} - aT_{n-1} + aT_n = T_{n+1}$$

so that all corresponding pairs are equal.

EXAMPLES. For $a = 4$, $T_3 = 4T_2 - 3T_1$ yields many sequences governed by

$$T_{n+1} = 4T_n - 4T_{n-1} + T_{n-2}$$

$$\dots 233, 89, 34, 13, 5, 2, 1, 1, 2, 5, 13, 34, 89, 233, \dots$$

$$\dots 177, 67, 25, 9, 3, 1, 1, 3, 9, 25, 67, 177, \dots$$

$$\dots 265, 100, 37, 13, 4, 1, 1, 4, 13, 37, 100, 265, \dots$$

$$\text{Case II.} \quad T_{n+1} = aT_n + bT_{n-1} - T_{n-2}, \quad T_{n-2} = aT_n + bT_{n-1} - T_{n+1}$$

A. Zero Term

$$(11) \quad T_0 = aT_2 + bT_1 - T_3, \quad T_{-1} = T_1 = aT_1 + bT_0 - T_2 = aT_1 + baT_2 + b^2T_1 - bT_3 - T_2$$

$$(a+b^2-1)T_1 + (ba-1)T_2 - bT_3 = 0$$

$$(12) \quad T_{-2} = T_2 = aT_0 - bT_{-1} - T_1 = a^2T_2 + abT_1 - aT_3 + bT_{-1} - T_1$$

$$(ab+b-1)T_1 + (a^2-1)T_2 - aT_3 = 0$$

$$(13) \quad T_{-3} = T_3 = aT_1 + bT_2 - aT_2 - bT_1 + T_3$$

$$(a-b)(T_1 - T_2) = 0$$

so that $b = a$ satisfies this relation.

Equations (11) and (12) both become for $b = a$:

$$(14) \quad (a^2 + a - 1)T_1 + (a^2 - 1)T_2 - aT_3 = 0.$$

For $a = 2$, $2T_3 = 5T_1 + 3T_2$ yields an infinity of sequences satisfying

$$\begin{aligned} T_{n+1} &= 2T_n + 2T_{n-1} - T_{n-2} \\ \dots 64, 25, 9, 4, 1, 1, 0, 1, 1, 4, 9, 25, 64, \dots \\ \dots 129, 49, 19, 7, 3, 1, 1, 1, 3, 7, 19, 49, 129, \dots \\ \dots 194, 73, 29, 10, 5, 1, 2, 1, 5, 10, 29, 73, 194, \dots \\ \dots 259, 97, 39, 13, 7, 1, 3, 1, 7, 13, 39, 97, 259, \dots \end{aligned}$$

B. No Zero Term

$$(15) \quad T_{n-2} = T_{n+1} - aT_n - bT_{n-1}, \quad T_{-1} = T_3 - aT_2 - bT_1$$

$$(b+1)T_1 + aT_2 = T_3$$

$$T_{-2} = T_2 = T_2 - aT_1 - bT_{-1}$$

$$(16) \quad (a+b)T_1 = 0.$$

Equation (15) becomes $T_3 = (1-a)T_1 + aT_2$ for $b = -a$. Now, $T_{-3} = T_3 = T_1 - aT_{-1} - bT_{-2}$

$$(17) \quad T_3 = (1-a)T_1 + aT_2$$

in agreement with (15) if $b = -a$.

$$T_{-4} = T_{-1} - aT_{-2} + aT_{-3} = aT_3 - aT_2 + T_1$$

whereas

$$T_4 = aT_3 - aT_2 - T_1$$

so that $T_1 = 0$ if $T_{-4} = T_4$.

Similarly setting $T_{-5} = T_5$ makes $T_2 = 0$, etc. Hence this case yields nothing more than the trivial result $\dots 0, 0, 0, 0, \dots$.

FOURTH-ORDER SEQUENCES

Case I.

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + T_{n-3}$$

A. Zero Term

$$T_{n-3} = T_{n+1} - aT_n - bT_{n-1} - cT_{n-2}, \quad T_0 = T_4 - aT_3 - bT_2 - cT_1$$

$$(18) \quad \begin{aligned} T_{-1} = T_1 = T_3 - aT_2 - bT_1 - cT_0 = T_3 - aT_2 - bT_1 - cT_4 + acT_3 + bcT_2 + c^2T_1 \\ (c^2 - b - 1)T_1 + (bc - a)T_2 + (ac + 1)T_3 - cT_4 = 0 \end{aligned}$$

$$(19) \quad \begin{aligned} T_{-2} = T_2 = T_2 - aT_1 - bT_0 - cT_{-1} = T_2 - aT_1 - bT_4 + abT_3 + b^2T_2 + bcT_1 - cT_1 \\ (bc - c - a)T_1 + b^2T_2 + abT_3 - bT_4 = 0 \end{aligned}$$

$$(20) \quad \begin{aligned} T_{-3} = T_3 = T_1 - aT_0 - bT_{-1} - cT_{-2} = T_1 - aT_4 + a^2T_3 + abT_2 + acT_1 - bT_1 - cT_2 \\ (ac - b + 1)T_1 + (ab - c)T_2 + (a^2 - 1)T_3 - aT_4 = 0 \end{aligned}$$

$$(21) \quad \begin{aligned} T_{-4} = T_4 = T_0 - aT_{-1} - bT_{-2} - cT_{-3} = T_4 - aT_3 - bT_2 - cT_1 - aT_1 - bT_2 - cT_3 \\ (a+c)T_1 + 2bT_2 + (a+c)T_3 = 0. \end{aligned}$$

If this set of four equations in T_1, T_2, T_3, T_4 is to have a non-zero solution, the determinant of the coefficients must be zero.

$$\begin{vmatrix} c^2 - b - 1 & bc - a & ac + 1 & -c \\ bc - c - a & b^2 & ab & -b \\ ac - b + 1 & ab - c & a^2 - 1 & -a \\ a + c & 2b & a + c & 0 \end{vmatrix} = 0$$

from which

$$(22) \quad (a+b+c)(-a+b-c)(a^2 - c^2 + 4b) = 0.$$

Before proceeding to further analysis some relations will be derived from equations (18) to (20). From (18) and (19)

$$(23) \quad (c^2 + ac - b^2 - b)T_1 - abT_2 + bT_3 = 0.$$

From (19) and (20)

$$(24) \quad (b^2 - b - ac - a^2)T_1 + bcT_2 + bT_3 = 0$$

and from (23) and (24)

$$(25) \quad (c^2 + a^2 + 2ac - 2b^2)T_1 = b(a+c)T_2.$$

THE CONDITION $a + b + c = 0$

$b = -a - c$ substituted into (25) gives

$$(c^2 + a^2 + 2ac - 2c^2 - 2a^2 - 4ac)T_1 = -(a+c)^2 T_2$$

so that $T_1 = T_2$. Then by (21)

$$(a+c)T_1 + 2(-a-c)T_1 + (a+c)T_3 = 0$$

so that $T_3 = T_1$. By (18),

$$(c^2 + a + c - 1 - c^2 - ac - a + ac + 1)T_1 = cT_4$$

so that $T_4 = T_1$. If the terms up to T_n are all equal to T_1 , then

$$T_{n+1} = aT_1 + (-a-c)T_1 + cT_1 + T_1 = T_1$$

so that all terms of the sequence are the same.

THE CONDITION $-a + b - c = 0$

$b = a + c$ leads to

$$T_2 = -T_1, \quad T_3 = T_1, \quad T_4 = -T_1.$$

If this alternation holds up to T_n , then

$$T_{n+1} = [a(-1)^{n-1} + (a+c)(-1)^n + c(-1)^{n-1} + (-1)^n]T_1 = (-1)^n T_1$$

so that the alternation continues.

THE CONDITION $a^2 - c^2 + 4b = 0$

a and c must be of the same parity.

EXAMPLE: $a = 1, \quad b = 12, \quad c = 7.$

Using Eqs. (18), (19) and (20) we obtain:

$$36T_1 + 83T_2 + 8T_3 - 7T_4 = 0, \quad 76T_1 + 144T_2 + 12T_3 - 12T_4 = 0, \quad -4T_1 + 5T_2 + 0T_3 - T_4 = 0.$$

from which $T_1 : T_2 : T_3 : T_4 = 3 : -7 : 18 : -47$.

Using the recursion relation

$$T_{n+1} = T_n + 12T_{n-1} + 7T_{n-2} + T_{n-3}$$

and a corresponding backward recursion relation, the following terms were obtained:

$$\dots 843, -322, 123, -47, 18, -7, 3, -2, 3, -7, 18, -47, 123, -322, 843, \dots$$

Second-Order Factor

If the symmetry is to continue beyond a term T_n , the condition for this would be:

$$T_{-n-1} = T_{n+1} = T_{-n+3} - aT_{-n+2} - bT_{-n+1} - cT_{-n} = T_{n-3} - aT_{n-2} - bT_{n-1} - cT_n.$$

But

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + T_{n-3}.$$

Hence there is a relation

$$(a+c)T_n + 2bT_{n-1} + (a+c)T_{n-2} = 0.$$

But since $4b = (c-a)/(c+a)$ we have in fact

$$T_n = (a-c)T_{n-1}/2 - T_{n-2}.$$

Thus if the symmetry is to continue the terms must satisfy a second-order recursion relation. That they do so can be seen from factoring

$x^4 - ax^3 - bx - c - 1 = 0$ into factors $(x^2 + Ex + 1)(x^2 + Fx - 1) = 0$,
where E is $(c - a)/2$. The conditions would be:

$$(c - a)/2 + F = -a \quad \text{or} \quad F = -(a + c)/2$$

from the coefficient of x cubed and the same value of F comes from the coefficient of x . Then the coefficient of x^2 would be:

$$EF = (-c^2 + a^2)/4 = -b$$

as required. Hence the terms obey this second-order relation and this insures the continuation of symmetry beyond T_{-4} . Note that this is not a proper fourth-order symmetric sequence.

B. No Zero Term

$$(26) \quad T_{n-3} = T_{n+1} - aT_n - bT_{n-1} - cT_{n-2}, \quad T_{-1} = T_1 = T_4 - aT_3 - bT_2 - cT_1$$

$$(c+1)T_1 + bT_2 + aT_3 - T_4 = 0$$

$$(27) \quad T_{-2} = T_2 = T_3 - aT_2 - bT_1 - cT_{-1}$$

$$(b+c)T_1 + (a+1)T_2 - T_3 = 0$$

$$(28) \quad T_{-3} = T_3 = T_2 - aT_1 - bT_{-1} - cT_{-2}$$

$$(a+b)T_1 + (c-1)T_2 + T_3 = 0$$

$$(29) \quad T_{-4} = T_4 = T_1 - aT_{-1} - bT_{-2} - cT_{-3}$$

$$(a-1)T_1 + bT_2 + cT_3 + T_4 = 0.$$

To have a non-zero solution the following determinant must be zero.

$$\begin{vmatrix} c+1 & b & a & -1 \\ b+c & a+1 & -1 & 0 \\ a+b & c-1 & 1 & 0 \\ a-1 & b & c & 1 \end{vmatrix} = 0$$

or

$$(30) \quad (a+b+c)(c^2 - a^2 - 4b) = 0.$$

As in the zero case, the condition $a+b+c=0$ leads to a sequence where all terms are the same. The other condition requires that the fourth-order recursion relation have a second-order factor which the terms of the symmetric sequence must obey. Hence this is a degenerate case also.

Case II.

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}$$

A. Zero Term

$$T_{n-3} = aT_n + bT_{n-1} + cT_{n-2} - T_{n+1}$$

If the symmetry is to continue indefinitely

$$T_{-n-1} = aT_{-n+2} + bT_{-n+1} + cT_{-n} - T_{-n+3}$$

$$T_{n+1} = aT_{n-2} + bT_{n-1} + cT_n - T_{n-3} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}$$

$$(a-c)(T_{n-2} - T_n) = 0$$

so that $a=c$ unless there is to be a recursion relation of lower order.

$$T_0 = aT_3 + bT_2 + aT_1 - T_4, \quad T_{-1} = T_1 = aT_2 + bT_1 + aT_0 - T_3$$

from which

$$(31) \quad (a^2 + b - 1)T_1 + a(1+b)T_2 + (a^2 - 1)T_3 = aT_4$$

from which

$$(32) \quad a(2+b)T_1 + (b^2 - 2)T_2 + abT_3 = bT_4.$$

Other relations simply repeat one of the above. Eliminating T_4 from (31) and (32):

$$(33) \quad (b^2 - b - 2a^2)T_1 + a(b+2)T_2 - bT_3 = 0$$

For given a and b , a suitable selection of T_1 and T_2 will give an integral value for T_3 . Thus for $a=7, b=-5$,

$$-68T_1 - 21T_2 = -5T_3.$$

$$T_1 = 1, \quad T_2 = 2, \quad T_3 = 22.$$

Then from (31), $T_4 = 149$. The symmetric sequence:

$$\dots 38494, 6029, 946, 149, 22, 2, 1, 2, 1, 2, 22, 149, 946, 6029, 38494, \dots$$

is governed by the recursion relation:

$$T_{n+1} = 7T_n - 5T_{n-1} + 7T_{n-2} - T_{n-3}.$$

B. No Zero Term

As before the continuation of symmetry for all terms requires that $a=c$ in the relation

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}.$$

Two relations are obtained from the requirement $T_{-1} = T_1$ and $T_{-2} = T_2$, namely:

$$(34) \quad (a-1)T_1 + bT_2 + aT_3 = T_4$$

$$(35) \quad (b+a)T_1 + (a-1)T_2 = T_3$$

The relations for T_{-3} and T_{-4} repeat these in inverse order.

EXAMPLE: $a = -2, \quad b = 5, \quad -3T_1 - 3T_2 = T_3$

$$T_1 = 4, \quad T_2 = 7, \quad T_3 = -9.$$

Then from (34), $T_4 = 41$.

The symmetric sequence:

$$\dots 6399, -1810, 506, -145, 41, -9, 7, 4, 4, 7, -9, 41, -145, 506, -1810, 6399, \dots$$

obeys the recursion relation:

$$T_{n+1} = -2T_n + 5T_{n-1} - 2T_{n-2} - T_{n-3}$$

FIFTH-ORDER SEQUENCES

Case I.

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + dT_{n-3} + T_{n-4}$$

A. Zero Term

To insure symmetry for all n we set:

$$T_{-n-1} = T_{n+1} = T_{-n+4} - aT_{-n+3} - bT_{-n+2} - cT_{-n+1} - dT_{-n} = T_{n-4} - aT_{n-3} - bT_{n-2} - cT_{n-1} - dT_n.$$

Combining this with the original recursion relation:

$$(a+d)(T_n + T_{n-3}) + (b+c)(T_{n-1} + T_{n-2}) = 0$$

so that $d = -a$ and $b = -c$ are necessary conditions to prevent reduction to a lower order recurrence relation.

Using the same techniques as previously we have the relations:

$$(36) \quad (a^2 + b - 1)T_1 + (ab - b)T_2 + (-ab - a)T_3 + (1 - a^2)T_4 + aT_5 = 0$$

$$(37) \quad (ab - b + a)T_1 + (b^2 - a - 1)T_2 + (1 - b^2)T_3 - abT_4 + bT_5 = 0.$$

Eliminating T_5 from (36) and (37) gives:

$$(38) \quad (b^2 - b + ab - a^2)T_1 + (a^2 + a - b^2)T_2 + (-ab - a)T_3 + bT_4 = 0.$$

EXAMPLE: $a = 5, \quad b = -3$ from which

$$-28T_1 + 21T_2 + 10T_3 = 3T_4$$

which is satisfied by $T_1 = 1, T_2 = 3, T_3 = 4, T_4 = 25$. Then from (36)

$$21T_1 - 12T_2 + 10T_3 - 24T_4 = -5T_5$$

which gives $T_5 = 115$.

The sequence

... 190299, 43060, 9745, 2203, 498, 115, 25, 4, 3, 1, -2, 1, 3, 4, 25, 115, 498, 2203, 9745, 43060, 190299, ...

is governed by the recursion relation:

$$T_{n+1} = 5T_n - 3T_{n-1} + 3T_{n-2} - 5T_{n-3} + T_{n-4} .$$

B. No Zero Term

An entirely similar analysis leads to two relations:

$$(39) \quad T_5 = (1-a)T_1 - bT_2 + bT_3 + aT_4$$

$$(40) \quad T_4 = (-b-a)T_1 + (b+1)T_2 + aT_3$$

EXAMPLE. $a = 5$, $b = -3$. From (40),

$$T_4 = -2T_1 - 2T_2 + 5T_3$$

which is satisfied by $T_1 = 1$, $T_2 = 3$, $T_3 = 4$, $T_4 = 12$.

Then by (39), $T_5 = -4T_1 + 3T_2 - 3T_3 + 5T_4 = 53$. The sequence

... 19428, 4397, 995, 227, 53, 12, 4, 3, 1, 1, 3, 4, 12, 53, 227, 995, 4397, 19428, ...

is governed by the recursion relation:

$$T_{n+1} = 5T_n - 3T_{n-1} + 3T_{n-2} - 5T_{n-3} + T_{n-4} .$$

Case II.

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + dT_{n-3} - T_{n-4} .$$

In this case symmetry in the sequence requires that $a = d$ and $b = c$.

A. Zero Case

The final relations obtained from the analysis are:

$$(41) \quad (a^2 + b - 1)T_1 + (ab + b)T_2 + (ab + a)T_3 + (a^2 - 1)T_4 = aT_5$$

$$(42) \quad (ab + a + b)T_1 + (b^2 + a - 1)T_2 + (b^2 - 1)T_3 + abT_4 = bT_5$$

from which

$$(43) \quad (b^2 - b - a^2 - ab)T_1 + (b^2 - a^2 + a)T_2 + (ab + a)T_3 = bT_4 .$$

EXAMPLE. $a = 3$, $b = -7$. (43) becomes

$$68T_1 + 43T_2 - 18T_3 = -7T_4$$

which is satisfied by

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 9, \quad T_4 = -5 .$$

Then from (41),

$$T_1 - 28T_2 - 18T_3 + 8T_4 = 3T_5 \quad \text{gives} \quad T_5 = -95 .$$

The symmetric sequence:

... 2203, -191, -305, -95, -5, 9, 3, 1, -1, 1, 3, 9, -5, -95, -305, -191, 2203, ...

is governed by the recursion relation:

$$T_{n+1} = 3T_n - 7T_{n-1} - 7T_{n-2} + 3T_{n-3} - T_{n-4}$$

B. No Zero Term

The relations obtained are:

$$(44) \quad (a-1)T_1 + bT_2 + bT_3 + aT_4 = T_5$$

$$(45) \quad (a+b)T_1 + (b-1)T_2 + aT_3 = T_4$$

$$(46) \quad bT_1 + aT_2 = T_3$$

EXAMPLE. $a = -5$, $b = 7$. (46) becomes $7T_1 - 5T_2 = T_3$ which is satisfied by

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = -8 .$$

Then (45)

$$2T_1 + 6T_2 - 5T_3 = T_4 \quad \text{gives} \quad T_4 = 60 . .$$

Finally (44)

$$-6T_1 + 7T_2 + 7T_3 - 5T_4 = T_5$$

gives a value $T_5 = -341$. The symmetric sequence:

$$\dots 72667, -12195, 2053, -341, 60, -8, 3, 1, 1, 3, -8, 60, -341, 2053, -12195, 72667, \dots$$

is governed by the recursion relation:

$$T_{n+1} = -5T_n + 7T_{n-1} + 7T_{n-2} - 5T_{n-3} - T_{n-4}.$$

CONCLUSION

From this investigation the following general approach to creating symmetric sequences of integers governed by linear recursion relations emerges.

(1) Given a linear recursion relation of order k ,

$$T_{n+1} = a_1T_n + a_2T_{n-1} + \dots + a_{k-1}T_{n-k+2} + T_{n-k+1}$$

the condition of symmetry in the sequence requires that:

$$a_j = -a_{k-j}$$

and for the recursion relation:

$$T_{n+1} = a_1T_n + a_2T_{n-1} + \dots + a_{k-1}T_{n-k+2} - T_{n-k+1}$$

symmetry requires that $a_j = a_{k-j}$.

(2) For the reduced number of parameters a_i , set up a corresponding number of symmetry conditions using the first few terms of the sequence.

(3) Using these conditions, select values for the parameters a_i and then find starting values in integers that satisfy the given conditions.
