NEW RELATIONS BETWEEN FIBONACCI AND BERNOULLI NUMBERS

PAUL F. BYRD
San Jose State University, San Jose, California 95192

1. INTRODUCTION

There seems to be no end to the number or variety of identities involving the Fibonacci sequence and/or its relatives. During the past decade, hundreds of such relations have been published in this journal alone. Those interesting identities, however, are mostly "pure"—containing terms within the same family; that is, not many of them are relations that involve a Fibonacci-type sequence together with some *other* classical sequence having different properties.

The family of Fibonacci-like numbers, for example, satisfies simple recurrence relations with constant coefficients while such famous sequences as those of Bernoulli satisfy more complicated difference equations having variable coefficients. It is thus of interest to pursue the questions: Can these sequences nevertheless be expressed simply in terms of each other? What kinds of identities can one easily find that involve both of them, etc.? Some relations answering such questions have been developed by Gould in [6] and by Kelisky in [8].

This article gives further answers in a systematic way with the use of several simple techniques. The paper will present various explicit relations between Fibonacci numbers and the number sequences of Bernoulli.[1] Relations involving the *generalized* Bernoulli numbers will represent a one-parameter, infinite class of such identities. Little detailed discussion, however, is given of the many special properties of the Bernoulli numbers themselves, for they have been the object of much published research for two hundred years.

2. BACKGROUND PRELIMINARIES

BERNOULLI POLYNOMIALS AND BERNOULLI NUMBERS

We begin by reviewing some prpperties of Bernoulli numbers and polynomials that will be needed for our purpose. The *Bernoulli polynomials* $B_n(x)$ of the n^{th} degree and *first order*[2] may be defined by the exponential generating function

$$(1)\qquad \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad |t| < 2\pi .$$

(See, for instance, [4] and [10].) More explicitly, these polynomials are given by the equation

$$(2)\qquad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

where B_k are the so-called *Bernoulli numbers.* One definition of the Bernoulli number sequence[3]

$$\left\{ 1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, 5/66, \right\} \cdots$$

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[1] A subsequent paper will explicitly relate the Fibonacci and Lucas sequences to the famous numbers of Euler.

[2] Bernoulli numbers of *higher order* will be defined later.

[3] Rather than $B_n(0)$, some authors prefer to call b_n the ordinary Bernoulli numbers, where $b_n = (-1)^{n+1}B_{2n}$, $n > 1$. The numbers b_n are essentially the absolute values of the non-zero elements in the B_n sequence. All the numbers are *rational;* they have applications in several branches of mathematics, appearing in the theory of numbers in the remarkable theorem of von Staudt-Clausen. (See, for example, [2], [3], and [5].)

is

$$B_k \equiv B_k(0). \tag{3}$$

Alternately, the numbers B_k may be defined by means of the generating formula

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad |t| < 2\pi . \tag{4}$$

Using combinatorial techniques given by Riordan in [11], one can *invert* Eq. (2) to obtain

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k(x)}{n-k+1} . \tag{5}$$

It can also be shown that special values of $B_n(x)$ are

$$\begin{cases} B_n(0) = (-1)^n B_n(1) = B_n, \; n = 0, 1, 2, \cdots \\ B_1(0) = B_1(1) - 1, \; B_0 = 1 \\ B_{2n+1}(0) = 0, \; n = 1, 2, \cdots \end{cases} \tag{6}$$

and that $B_1 = -\frac{1}{2}$ is the only non-zero Bernoulli number with odd index. We can thus write (2) as

$$B_n(x) = x^n - \frac{n}{2}x^{n-1} + \sum_{k=1}^{[n/2]} \binom{n}{2k} B_{2k} x^{n-2k} . \tag{7}$$

The $(2k)^{th}$ Bernoulli number is computed by means of the recurrence relation

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} \sum_{m=0}^{k-1} \binom{2k+1}{2m} B_{2m}, \quad k \geq 1 \tag{8}$$

with $B_0 = 1$, or explicitly by use of the little-known formula

$$B_{2k} = \sum_{n=0}^{2k} \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} j^{2k}, \quad k \geq 0. \tag{9}$$

With this finite sum substituted into (7), it is possible to express the Bernoulli polynomials in a closed form not involving the Bernoulli numbers themselves. In fact (see [7]),

$$B_k(x) = \sum_{n=0}^{k} \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x+j)^k .$$

FIBONACCI POLYNOMIALS AND FIBONACCI NUMBERS

We recall that the *Fibonacci polynomials* $F_n(x)$ of degree $(n-1)$ are solutions of the recurrence relation

$$F_{k+1}(x) = xF_k(x) + F_{k-1}(x), \quad k \geq 1 \tag{10}$$

with $F_1(x) = 1$ and $F_2(x) = x$. More explicitly, we have

$$F_{k+1}(x) = \sum_{m=0}^{[k/2]} \binom{k-m}{m} x^{k-2m} , \tag{11}$$

and note that the numbers

$$F_{k+1}(1) \equiv F_k \tag{12}$$

are the *Fibonacci numbers.* These numbers, and their closest relative, the *Lucas numbers* L_n, are often defined by the familiar *generating functions*

(13) $$\frac{e^{at}-e^{bt}}{\sqrt{5}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad e^{at}+e^{bt} = \sum_{n=0}^{\infty} L_n \frac{t^n}{n!},$$

or, in the so-called Binet forms, by the formulas

(14) $$F = \frac{a^n-b^n}{a-b}, \qquad L = a^n+b^n,$$

where

(15) $$a = (1+\sqrt{5})/2, \qquad b = (1-\sqrt{5})/2.$$

3. RELATIONS BETWEEN FIBONACCI AND BERNOULLI NUMBERS

With the above preliminaries, an explicit relation

(16) $$F_{2N+1}(x) = \sum_{k=0}^{2N} C_{k,N}B_k(x) \qquad N \geqslant 0$$

expressing the Fibonacci polynomials of even degree in terms of Bernoulli polynomials, can now be developed in the following simple way. Equation (11) gives

(17) $$F_{2N+1}(x) = \sum_{k=0}^{N} \binom{2N-n}{n} x^{2N-2n},$$

so with the inversion formula (5) inserted in (17), we have

$$F_{2N+1}(x) = \sum_{n=0}^{N} \binom{2N-n}{n} \sum_{k=0}^{2N-2n} \binom{2N-2n}{k} \frac{B_k(x)}{2N-2n-k+1},$$

or, on reversing order of summation,

(18) $$F_{2N+1}(x) = \sum_{k=0}^{2N} B_k(x) \sum_{n=0}^{\left[\frac{2N-k}{2}\right]} \binom{2N-n}{n}\binom{2N-2n}{k} \frac{1}{2N-2n-k+1}.$$

Thus, with coefficients $C_{k,N}$ given by

(19) $$C_{k,N} = \sum_{n=0}^{\left[\frac{2N-k}{2}\right]} \binom{2N-n}{n}\binom{2N-2n}{k} \frac{1}{2N-2n-k+1},$$

we have the desired relation

(20) $$F_{2N+1}(x) = \sum_{k=0}^{2N} C_{k,N}B_k(x).$$

Similarly, for Fibonacci polynomials $F_{2N+2}(x)$ of odd degree, it is easy to show that expressed by

(21) $$F_{2N+2}(x) = \sum_{k=0}^{2N+1} A_{k,N}B_k(x)$$

with coefficients

(22) $$A_{k,N} = \sum_{n=0}^{\left[\frac{2N+1-k}{2}\right]} \binom{2N+1-n}{n}\binom{2N+1-2n}{k} \frac{1}{2N-2n-k+2}.$$

Since $F_n(1) \equiv F_n$, and

$$(23)\qquad \begin{cases} B_k(1) = B_k(0) = B_k, & (k \geq 2) \\ B_{2m+1}(1) \equiv B_{2m+1}(0) \equiv B_{2m+1} = 0, & (m \geq 1) \end{cases}$$

the equations (20) and (21) will immediately furnish explicit relations which express Fibonacci numbers in terms of Bernoulli numbers. From (20), with $x = 1$, we thus have

$$(24)\qquad F_{2N+1} = C_{1,N}B_1(1) + \sum_{k=0}^{N} C_{2k,N}B_{2k}.$$

But, $B_1(1) = -B_1 = \tfrac{1}{2}$, and

$$(25)\qquad C_{1,N} = \sum_{n=0}^{N-1} \binom{2N-n}{n} = -1 + F_{2N+1}.$$

Hence,

$$(26)\qquad F_{2N+1} \equiv -1 + 2\sum_{k=0}^{N} C_{2k,N}B_{2k},$$

where

$$(27)\qquad C_{2k,N} = \sum_{n=0}^{N-k} \binom{2N-n}{n}\binom{2N-2n}{2k}\frac{1}{2N+1-2k-2n}.$$

With the same procedure, using (21) and (23), we find that

$$(28)\qquad F_{2N+2} = 2\sum_{k=0}^{N} A_{2k,N}B_{2k},$$

where

$$(29)\qquad A_{2k,N} = \sum_{n=0}^{N-k} \binom{2N+1-n}{n}\binom{2N+1-2n}{2k}\frac{1}{2N-2n-2k+2}.$$

Inverse relations (expressing the Bernoulli polynomials and numbers in terms of those of Fibonacci) are equally important. In [1], the author showed how an analytic function can be expanded in polynomials associated with Fibonacci numbers, so the details of carrying this out in the special case of Bernoulli polynomials will be left to the reader.

4. SOME NEW IDENTITIES

With a little inventive manipulation* and the application of Cauchy's rule for multiplying power series, many new relations between Fibonacci and Bernoulli numbers can be easily obtained. Although these are all special examples of the general case presented later, there may be an advantage to many readers of this Journal to consider them in some detail.

EXAMPLE 1. Starting with Eq. (13), we have

$$(30)\qquad e^{at} - e^{bt} = e^{bt}[e^{(a-b)t} - 1] = e^{bt}[e^{t\sqrt{5}} - 1] = \sqrt{5}\sum_{n=0}^{\infty} F_n \frac{t^n}{n!},$$

or

$$(31)\qquad te^{bt} = \frac{t\sqrt{5}}{e^{t\sqrt{5}} - 1}\sum_{n=0}^{\infty} F_n \frac{t^n}{n!}.$$

*Series manipulation has long been a most powerful fundamental tool for obtaining or operating with generating functions as we shall be doing throughout this article.

Expanding the left-hand side, and noting from (4) that

$$\frac{t\sqrt{5}}{e^{t\sqrt{5}}-1} = \sum_{n=0}^{\infty} B_n \frac{(t\sqrt{5})^n}{n!} \tag{32}$$

one sees that (31) becomes

$$t\sum_{n=0}^{\infty} b^n \frac{t^n}{n!} = \left[\sum_{s=0}^{\infty} B_s(\sqrt{5})^s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} F_n \frac{t^n}{n!}\right]. \tag{33}$$

If we make use of Cauchy's rule and equate coefficients, we find the identity

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k \frac{F_{n-k+1}}{n-k+1} B_k = b^n , \tag{34}$$

which holds for all $n \geqslant 0$. (It may appear simpler to use

$$\frac{1}{n-k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k} .$$

This can apply to some subsequent formulas presented here.)

EXAMPLE 2. On the other hand, if we write

$$e^{at} - e^{bt} = -e^{at}[e^{-t\sqrt{5}} - 1] = \sqrt{5}\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} , \tag{35}$$

we get

$$\sum_{n=0}^{\infty} a^n \frac{t^n}{n!} = \left[\sum_{s=0}^{\infty} (-\sqrt{5})^s B_s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} F_n \frac{t^{n-1}}{n!}\right] ,$$

and thus obtain, since $F_0 = 0$, the identity

$$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \frac{F_{n-k+1}}{n-k+1} B_k = a^n, \qquad n \geqslant 0. \tag{36}$$

EXAMPLE 3. Recalling that the Lucas numbers are given by

$$L_m = a^m + b^m, \qquad [a = (1+\sqrt{5})/2,\ b = (1-\sqrt{5})/2] , \tag{37}$$

one can add equations (34) and (36) to attain the more interesting identity

$$L_n = 2\sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k \frac{F_{n-2k+1}}{n-2k+1} B_{2k} , \qquad n \geqslant 0, \tag{38}$$

which contains three different number sequences—Lucas, Fibonacci, and Bernoulli. [However, subtracting (34) from (36) only gives the trivial identity $F_n = F_n$.]

5. AN EXTENSION

We now make a generalization involving Bernoulli numbers of *higher order.* Use of the same procedures just given will furnish a whole class of new identities.

DEFINITION

*Generalized Bernoulli numbers** $B_n^{(m)}$ of the m^{th} *order* are generated by the expansion

$$\frac{t^m}{(e^t-1)^m} = \sum_{n=0}^{\infty} B_n^{(m)} \frac{t^n}{n!}, \qquad |t| < 2\pi , \tag{40}$$

where m is any positive or negative integer. (If $m = 1$, one writes $B_n^{(m)} \equiv B_n$, omitting the superscript as we did before.) Thus,

*A thorough discussion of the properties of these numbers is given in [9].

(41) $$B_n^{(m)} = \frac{d^n}{dt^n}\left[\left(\frac{t}{e^t-1}\right)^m\right]_{t=0}$$

and we obtain the number sequence

(42) $$B_0^{(m)} = 1,\quad B_1^{(m)} = -\tfrac{1}{2}m,\quad B_2^{(m)} = \frac{1}{12}m(3m-1),\quad B_3^{(m)} = -\frac{1}{8}m^2(m-1),$$
$$B_4^{(m)} = \frac{1}{240}m(15m^3-30m^2+5m+2),\ \cdots .$$

The sequence satisfies the *partial difference equation*

(43) $$mB_n^{(m+1)} - (m-n)B_n^{(m)} + mnB_{n-1}^{(m)} = 0 .$$

If m is a negative integer, i.e., $m = -p,\ p \geqslant 1$, an explicit formula for the numbers is given by

(44) $$B_n^{(-p)} = \frac{n!}{(n+p)!}\sum_{r=0}^{p}(-1)^r\binom{p}{r}(p-r)^{n+p} .$$

SPECIAL CASE OF SECOND ORDER

Let us first consider the case when $m = 2$, and quickly obtain four new identities expressed in Eqs. (47)–(50) below. Note that

$$(e^{at}-e^{bt})^2 = e^{2bt}(e^{t\sqrt{5}}-1)^2$$

and also, using (13) that

(45) $$(e^{at}-e^{bt})^2 = [e^{2at}+e^{2bt}] - 2e^t = \sum_{n=0}^{\infty}[2^nL_n-2]\frac{t^n}{n!},$$

where L_n are again the Lucas numbers. One may thus write

(46) $$t^2e^{2bt} = \frac{5t^2}{(e^{t\sqrt{5}}-1)^2}\cdot\frac{1}{5}\sum_{n=0}^{\infty}[2^nL_n-2]\frac{t^n}{n!} .$$

Since, from Eq. (40),

$$\frac{(t\sqrt{5})^2}{(e^{t\sqrt{5}}-1)^2} = \sum_{n=0}^{\infty}(\sqrt{5})^nB_n^{(2)}\frac{t^n}{n!},$$

and since

$$e^{2bt} = \sum_{n=0}^{\infty}(2b)^n\frac{t^n}{n!},$$

relation (46) gives

$$5\sum_{n=0}^{\infty}(2b)^n\frac{t^n}{n!} = \left[\sum_{s=0}^{\infty}[2^sL_s-2]\frac{t^{s-2}}{s!}\right]\left[\sum_{n=0}^{\infty}(\sqrt{5})^nB_n^{(2)}\frac{t^n}{n!}\right]$$

We note that $2^sL_s - 2 = 0$ for $s = 0$ and $s = 1$, and we then use Cauchy's rule. For each value of $n \geqslant 0$, there results the identity

(47) $$\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k\frac{[2^{n-k+2}L_{n-k+2}-2]}{(n-k+1)(n-k+2)}B_k^{(2)} = 5(2b)^n$$

involving Lucas numbers and Bernoulli numbers of the second order.*

On the other hand, taking

$$(e^{at}-e^{bt})^2 \quad\text{as}\quad e^{2at}(e^{-t\sqrt{5}}-1)^2$$

leads to the identity

*The Bernoulli number sequence of order 2 is $\left\{1, -1, 5/6, -1/2, 1/10, -1/6, \cdots\right\}$.

$$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \frac{[2^{n-k+2}L_{n-k+2}-2]}{(n-k+1)(n-k+2)} B_k^{(2)} = 5(2a)^n \,. \tag{48}$$

If Eqs. (47) and (48) are added, one obtains the identity

$$L_n = \frac{2}{5(2)^n} \sum_{k=0}^{[n/2]} \binom{n}{2k} \frac{[2^{n-2k+2}L_{n-2k+2}-2]}{(n-2k+1)(n-2k+2)} 5^k B_{2k}^{(2)} \,, \tag{49}$$

while subtraction yields the identity

$$F_n = \frac{2}{5(2)^n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} 5^k \frac{2-2^{n-2k+1}L_{n-2k+1}}{(n-2k)(n-2k+1)} B_{2k+1}^{(2)} \,, \tag{50}$$

both relations being valid for all $n > 0$.

SPECIAL CASE OF NEGATIVE ORDER

Before discussing the most general case, let us take $m = -2$. Now, from (40), it is seen that

$$(t\sqrt{5})^{-2}(e^{t\sqrt{5}}-1)^2 = \sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-2)} \frac{t^n}{n!} \,.$$

Thus

$$(e^{at}-e^{bt})^2 = [(t\sqrt{5})^2 e^{2bt}][(t\sqrt{5})^{-2}(e^{t\sqrt{5}}-1)^2] = (t\sqrt{5})^2 \left[\sum_{s=0}^{\infty} (2b)^s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-2)} \frac{t^n}{n!}\right].$$

On the other hand, in view of (13), we have

$$(e^{at}-e^{bt})^2 = [e^{2at}+e^{2bt}] - 2e^t = \sum_{n=0}^{\infty} [2^n L_n - 2] \frac{t^n}{n!} \,,$$

and therefore,

$$\frac{1}{5}\sum_{n=0}^{\infty} [2^{n+2}L_{n+2}-2] \frac{t^n}{(n+2)} = \left[\sum_{s=0}^{\infty} (2b)^s \frac{t^s}{s!}\right]\left[\sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-2)} \frac{t^n}{n!}\right].$$

From this equation there immediately results the identity

$$L_{n+2} = \frac{1}{2^{n+2}}\left[2 + 5(n+1)(n+2)\sum_{k=0}^{n} \binom{n}{k} (2b)^{n-k}(\sqrt{5})^k B_k^{(-2)}\right], \quad n \geq 0 \,. \tag{51}$$

Similarly, starting with

$$(e^{at}-e^{bt})^2 = [(t\sqrt{5})^2 e^{2at}][(t\sqrt{5})^{-2}(e^{-t\sqrt{5}}-1)^2] \,,$$

we are led to the identity

$$L_{n+2} = \frac{1}{2^{n+2}}\left[2 + 5(n+1)(n+2)\sum_{k=0}^{n} \binom{n}{k} (2a)^{n-k}(-1)^k(\sqrt{5})^k B_k^{(-2)}\right]. \tag{52}$$

If Eq. (51) is subtracted from (52), one obtains the identity

$$5\sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k B_{2k}^{(-2)} F_{n-2k} = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} 5^k B_{2k-1}^{(-2)} L_{n-2k-1} \,, \quad n \geq 1 \tag{53}$$

which involves Fibonacci numbers, Lucas numbers, and Bernoulli numbers of negative second order.

GENERAL CASE WHEN m IS AN ARBITRARY NEGATIVE INTEGER

Let $m = -p$ with p being a positive integer. Notice that

$$(54)\qquad (e^{at}-e^{bt})^p = [e^{pat}+(-1)^p e^{pbt}] + \sum_{r=1}^{p-1} (-1)^r \binom{p}{r} e^{[pa+(b-a)r]t}$$

and that

$$[e^{pat}+(-1)^p e^{pbt}] = \sum_{n=0}^{\infty} p^n L_n \frac{t^n}{n!} \qquad \text{if } p \text{ is even,}$$

$$= \sqrt{5}\sum_{n=0}^{\infty} p^n F_n \frac{t^n}{n!} \qquad \text{if } p \text{ is odd.}$$

It is also clear that

$$(55)\quad (e^{at}-e^{bt})^p = [(t\sqrt{5})^p e^{pbt}][(t\sqrt{5})^{-p}(e^{t\sqrt{5}}-1)^p] = (t\sqrt{5})^p \left[\sum_{r=0}^{\infty} (pb)^r \frac{t^r}{r!}\right]\left[\sum_{n=0}^{\infty} (\sqrt{5})^n B_n^{(-p)} \frac{t^n}{n!}\right].$$

Equating (54) and (55) results in the following two identities:

$$(56)\quad L_{n+p} = \frac{1}{p^{n+p}}\left\{-\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p} + (\sqrt{5})^p\frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pb)^{n-k}(\sqrt{5})^k B_k^{(-p)}\right\}$$

if p is even, and

$$(57)\quad F_{n+p} = \frac{1}{p^{n+p}}\left\{-\frac{1}{\sqrt{5}}\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p} + (\sqrt{5})^{p-1}\frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pb)^{n-k}(\sqrt{5})^k B_k^{(-p)}\right\}$$

when p is odd. If $p = 1$, the first summation does not appear, and (57) reduces to

$$(58)\qquad F_{n+1} = (n+1)\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(\sqrt{5})^k B_k^{(-1)} .$$

In all these formulas

$$a = (1+\sqrt{5})/2, \quad b = (1-\sqrt{5})/2, \quad \text{and} \quad b-a = -\sqrt{5} .$$

The identities (56) and (57) give new relations for each p, and thus represent a whole class of identities.

Another infinite class of such relations is obtained by beginning with

$$(e^{at}-e^{bt})^p = (-1)^p e^{pat}(e^{-t\sqrt{5}}-1)^p = (-1)^p[(t\sqrt{5})^p e^{pat}][(t\sqrt{5})^{-p}(e^{-t\sqrt{5}}-1)^p]$$

instead of with (55). This consideration yields

$$(59)\qquad L_{n+p} = \frac{1}{p^{n+p}}\left\{-\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p}\right.$$

$$\left. + (\sqrt{5})^p\frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pa)^{n-k}(-1)^k(\sqrt{5})^k B_k^{(-p)}\right\}$$

if p is even, and

$$(60)\qquad F_{n+p} = \frac{1}{p^{n+p}}\left\{-\frac{1}{\sqrt{5}}\sum_{r=1}^{p-1}(-1)^r\binom{p}{r}[pa+(b-a)r]^{n+p}\right.$$

$$\left. + (\sqrt{5})^{p-1}\frac{(n+p)!}{n!}\sum_{k=0}^{n}\binom{n}{k}(pa)^{n-k}(-1)^k(\sqrt{5})^k B_k^{(-p)}\right\}$$

when p is odd. For $p = 1$, (60) reduces to

$$F_{n+1} = (n+1)\sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k}(\sqrt{5})^k B_k^{(-1)} . \tag{61}$$

Subtracting relation (56) from (59) yields

$$5\sum_{k=0}^{[n/2]} \binom{n}{2k} p^{n-2k}5^k F_{n-2k}B_{2k}^{(-p)} = \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} p^{n-2k+1}5^k L_{n-2k+1}B_{2k-1}^{(-p)} \tag{62}$$

while subtracting (57) from (60) gives the same thing. Thus the identity (62) holds for all non-negative p, and for $n \geqslant 1$.

GENERAL CASE WHEN m IS AN ARBITRARY POSITIVE INTEGER

The same techniques of a little creative manipulation and the application of Cauchy's rule is used here. Without giving the details of the development, we shall just present the results.

For *even* positive values of m, one obtains the identities

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k \left\{ \frac{m^{n-k+m}L_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m}}{(n-k+m)!} \right\} (n-k)!B_k^{(m)} = (\sqrt{5})^m (mb)^n, \qquad n \geqslant 0 \tag{63}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \left\{ \frac{m^{n-k+m}L_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m}}{(n-k+m)!} \right\} (n-k)!B_k^{(m)} = (\sqrt{5})^m (ma)^n , \tag{64}$$

Adding these two identities yields

$$L_n = \frac{2}{(\sqrt{5})^m m^n} \sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k \frac{(n-2k)!}{(n-2k+m)!} \left\{ m^{n-2k+m}L_{n-2k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+m} \right\} B_{2k}^{(m)} , \qquad n \geqslant 0 \tag{65}$$

while subtraction gives

$$F_n = \frac{-2}{(\sqrt{5})^m m^n} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} 5^{k-1} \frac{(n-2k+1)!}{(n+2k+1+m)!} \left\{ m^{n-2k+1+m}L_{n-2k+1+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+1+m} \right\} B_{2k-1}^{(m)} , \qquad n \geqslant 1. \tag{66}$$

For *odd* positive values of m, there result the identities

$$\sum_{k=0}^{n} \binom{n}{k} (\sqrt{5})^k \frac{(n-k)!}{(n-k+m)!} \left\{ \sqrt{5}m^{n-k+m}F_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m} \right\} B_k^{(m)} = (\sqrt{5})^m (mb)^n , \qquad n \geqslant 0, \tag{67}$$

and

$$(68)\quad \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \frac{(n-k)!}{(n-k+m)!} \left\{ \sqrt{5}m^{n-k+m} F_{n-k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-k+m} \right\} B_k^{(m)} = (\sqrt{5})^m (ma)^n, \qquad n \geq 0.$$

If (67) and (68) are added or subtracted, we get, respectively,

$$(69)\quad L_n = \frac{2}{(\sqrt{5})^m m^n} \sum_{k=0}^{[n/2]} \binom{n}{2k} 5^k \left\{ \sqrt{5}m^{n-2k+m} F_{n-2k+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+m} \right\} \frac{(n-2k)!}{(n-2k+m)!} B_{2k}^{(m)} \qquad n \geq 0$$

and

$$(70)\quad F_n = \frac{-2}{(\sqrt{5})^m m^n} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} 5^{k-1} \left\{ \sqrt{5}m^{n-2k+1+m} F_{n-2k+1+m} + \sum_{r=1}^{m-1} (-1)^r \binom{m}{r} [ma+(b-a)r]^{n-2k+1+m} \right\} \frac{(n-2k+1)!}{(n-2k+1+m)!} B_{2k-1}^{(m)} \qquad n \geq 1$$

We note that the identities given by each of the above eight relations, involving Bernoulli numbers of positive order, constitute one-parameter infinite classes since a different identity results for each value of $m > 0$.

6. REMARKS

Making a direct connection of Stirling numbers of the second kind to Bernoulli generalized numbers permits one to immediately utilize some of the above results in order to find explicit relations between Stirling numbers and those of Fibonacci or Lucas.

Stirling numbers of the second kind $S(n,j)$, which represent the number of ways of partitioning a set of n elements into j non-empty subsets, are the coefficients in the expansion

$$(71)\quad x^n = \sum_{j=1}^{n} S(n,j)(x)_j\,,$$

where $(x)_j$ is the factorial polynomial

$$(72)\quad (x)_j = x(x-1)(x-2)\cdots(x-j+1).$$

(See, for example, [11].) Since these numbers are also defined by the generating function

$$(73)\quad (e^t-1)^m = m! \sum_{n=m}^{\infty} S(n,m)\frac{t^n}{n!}\,,$$

it is easy to show, in view of (40), that they are related to generalized Bernoulli numbers by the simple formula

$$(74)\quad \frac{(n+p)!}{n!}\binom{n}{k} B_k^{(-p)} = \binom{n+p}{k+p} S(k+p,p).$$

Substitution of this in to relations (56, (57), (59), (60), and (61) will immediately furnish identities involving Stirling numbers together with those of Fibonacci and Lucas. Although the resulting identities would essentially be the *same* (except for new notation or symbolism), they may nevertheless be interesting to those interested in Stirling numbers.

We have developed the identities in this article in a formal way without attempting to explore their implication or to find applications for them. Perhaps this paper will interest some reader to do so, as well as to make simplifications and further extensions. However, as interesting as such formulas may seem, one should pursue the more important question of whether or not they imply any new arithmetical properties, or more beautiful number theoretic theorems, of the various sequences involved.

It was pointed out by Zeitlin, a referee of this paper, that all the results here can be generalized to apply to sequences defined by

$$W_{n+2} = pW_{n+1} - qW_n$$

(See [12] for some properties of such sequences.)

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