FIBONACCI TILES

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1. INTRODUCTION

The conventional method of tiling the plane uses congruent geometric figures. That is, the plane is covered with non-overlapping translates of a given shape or tile [1]. Such tilings have interesting algebraic models in which the centers of each tile play an important role.

The plane can also be tiled with squares whose sides are in 1:1 correspondence with the Fibonacci numbers in the manner shown in Fig. 1 and such patterns can be used to demonstrate interesting algebraic properties of the Fibonacci numbers [2].

Similar spiral patterns can be obtained with squares whose sides are in 1:1 correspondence with similar recursive sequences of positive real numbers as in Fig. 2.

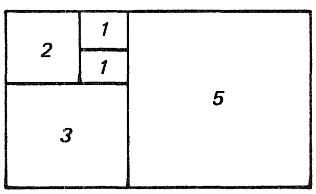


Figure 1

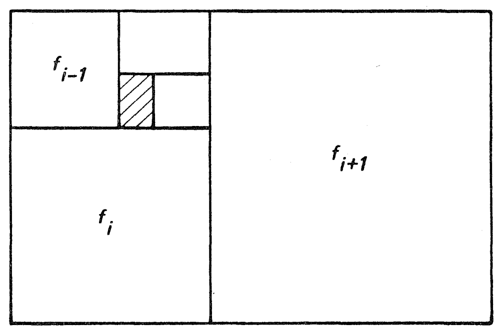


Figure 2

We will show that the centers of the squares in such a pattern all lie on two perpendicular straight lines and the slopes of these lines are independent of the choice of f_1 and f_2. Furthermore, the distances of the centers from the intersection of these two lines also form a recursive sequence.

2. CONSTRUCTION OF THE PATTERN

The pattern in Fig. 2 is a counter-clockwise spiral of squares which fills the plane except for a small initial rectangle. The side of the i^{th} square is denoted by f_i and the f_i are defined by

(1) $$f_{i+2} = f_{i+1} + f_i \quad \text{for} \quad i \geqslant 1 \quad \text{and} \quad 0 < f_1 \leqslant f_2 .$$

The side of the first square is f_1 and for notational convenience we define

$$f_i = f_{i+2} - f_{i+1} \quad \text{for} \quad i \leqslant 0 .$$

The position of successive squares in the spiral can be conveniently expressed in terms of an appropriate corner point of each square and a sequence of vectors which are parallel to the sides of the squares. Consider the sequence of vectors V_i defined by

$$V_1 = (1,0) \qquad V_{i+1} = V_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for} \quad i \geqslant 1 .$$

This sequence consists of four distinct vectors:

(2) $$V_i \in \{(1,0),(0,1),(-1,0),(0,-1)\} \quad .$$

The vestors in this sequence have the property that $V_{i+2} = -V_i$.

If P_1 denotes the lower right corner point of the first square (see Fig. 3) then successive corner points are given by

(3) $$P_i = P_{i-1} + f_{i+1}V_i \;.$$

The center C_i of the i^{th} square is obtained from the corresponding corner point (see Fig. 4) by means of the equation

(4) $$C_i = P_i + \frac{f_i}{2}\,(V_{i+1} - V_i)\;.$$

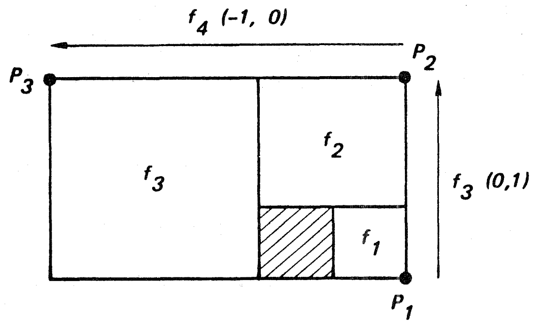


Figure 3

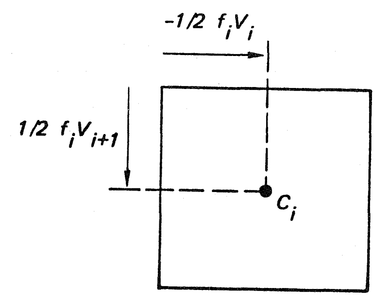


Figure 4

We now proceed to obtain an expression for the vector between alternate centers. Some sample values for P_i and C_i, are given in Tables 1 and 2.

TABLE 1

i	f_i	P_i	C_i	$d_i\sqrt{10}$
1	1	(1, −1)	(0.5, −0.5)	3
2	2	(1, 2)	(0, 1)	4
3	3	(−4,2)	(−2.5, 0.5)	7
4	5	(−4,−6)	(−1.5, −3.5)	11
5	8	(9, −6)	(5, −2)	18
6	13	(9, 15)	(2.5, 8.5)	29
7	21	(−25, 15)	(−14.5, 4.5)	47
8	34	(−25, 40)	(−8, −23)	76
9	55	(64, −40)	(36.5, −12.5)	123
10	89	(64, 104)	(19.5, 59.5)	199
11	144	(−169, 104)	(−97, 32)	322
12	233	(−169, −273)	(−52.5, −156.5)	521

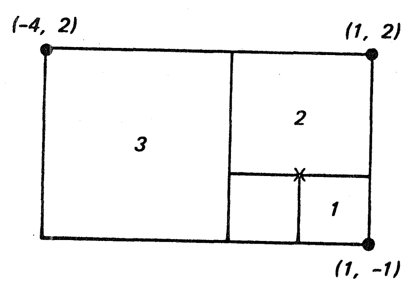
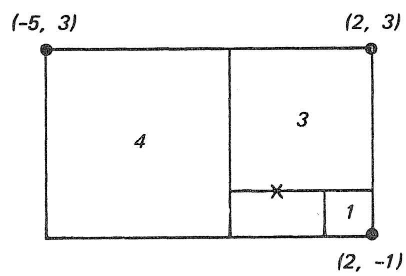


TABLE 2

i	f_i	P_i	C_i	$d_i\sqrt{10}$
1	1	(2, −1)	(1.5, −0.5)	5
2	3	(2, 3)	(0.5, 1.5)	10
3	4	(−5, 3)	(−3. 1)	15
4	7	(−5. 8)	(−1.5, −4.5)	25
4	11	(13, −8)	(7.5, −2.5)	40
6	18	(13, 21)	(4,12)	65
7	29	(−34, 21)	(−19.5, 6.5)	105
8	47	(−34, −55)	(−10.5, −31.5)	170
9	76	(89, −55)	(51, −17)	275
10	123	(89, 144)	(27.5, 82.5)	445
11	199	(−233, 144)	(−133.5, 44.5)	720
12	322	(−233, −377)	(−72, −216)	1165



3. STRUCTURAL PROPERTIES

Lemma 1.

$$C_i - C_{i-2} = \frac{f_{i-1}}{2}(3V_i - V_{i+1}) .$$

Proof. From Eq. (4), we have

$$C_i = P_i + \frac{f_i}{2}(V_{i+1} - V_i)$$

$$C_{i-2} = P_{i-2} + \frac{f_{i-2}}{2}(V_{i-1} - V_{i-2}) = P_{i-2} + \frac{f_{i-2}}{2}(V_i - V_{i+1})$$

(5) $$C_i - C_{i-2} = P_i - P_{i-2} + \frac{f_i}{2}(V_{i+1} - V_i) - \frac{f_{i-2}}{2}(V_i - V_{i+1}) .$$

Combining Eqs. (5) and (6) and collecting terms in V_i and V_{i+1} we have

$$C_i - C_{i-2} = \tfrac{1}{2}(2f_{i+1} - f_i - f_{i-2})V_i + \tfrac{1}{2}(f_{i-2} - f_i)V_{i+1} \quad .$$

Using the recursive definition of the f_i (see Eq. (1)), this reduces to

$$C_i - C_{i-2} = \frac{3f_{i-1}}{2}V_i - \frac{f_{i-1}}{2}V_{i+1} \quad .$$

Corollary 1.1. The distance between alternating centers is given by :

$$|C_i - C_{i-2}| = \frac{f_i\sqrt{10}}{2} \quad .$$

Proof. From the definition of the V_i we have

$$V_i \cdot V_i = 1 \qquad \text{and} \qquad V_i \cdot V_{i+1} = 0$$

$$|C_i - C_{i-2}|^2 = (C_i - C_{i-2})\cdot(C_i - C_{i-2}) = \frac{9}{4}f_{i-1}^2 + \frac{1}{4}f_{i-1}^2 = \frac{10}{4}f_{i-1}^2 .$$

Lemma 2. C_i, C_{i+2}, and C_{i+4} are colinear for all $i \geqslant 1$.

Proof. From Lemma 1 we have

$$C_{i+4} - C_{i+2} = \frac{f_{i+5}}{2}(3V_{i+4} - V_{i+5}) = -\frac{f_{i+5}}{2}(3V_{i+2} - V_{i+3}) = -\frac{f_{i+5}}{f_{i+3}}\cdot\frac{f_{i+3}}{2}(3V_{i+2} - V_{i+3}) = -\frac{f_{i+5}}{f_{i+3}}(C_{1+2} - C_i).$$

Hence $C_{i+4} - C_{i+2}$ is a multiple of $C_{i+2} - C_i$ and both vectors have the point C_{i+2} in common.

Theorem 1. The C_i all lie on two perpendicular straight lines. The slopes of these lines are 3 and $-(1/3)$ independent of the choice of f_1 and f_2.

Proof. By Lemma 2 we need only consider the slopes of $C_4 - C_2$ and $C_3 - C_1$.

$$C_4 - C_2 = \left(-\frac{f_3}{2}, -\frac{3f_3}{2}\right) \quad \text{and} \quad C_3 - C_1 = \left(-\frac{3f_2}{2}, \frac{f_2}{2}\right) .$$

Hence the slopes are 3 and $-(1/3)$.

Definition 1. Let I be the point of intersection for the two lines in Theorem 1, then the distance from C_i to I will be denoted by d_i. That is $d_i = |C_i - I|$. (Sample values are given in Tables 1 and 2.)

Lemma 3.

$$d_i + d_{i-2} = \frac{f_{i-1}\sqrt{10}}{2}, \quad d_i^2 + d_{i-1}^2 = \tfrac{1}{4}(f_{i+1}^2 + f_{i-2}^2) .$$

Proof. By the definition of d_i we have

$$d_i + d_{i-2} = |C_i - C_{i-2}|$$

and hence the first equation follows from Corollary 1.1.

From Equation 4, we have

$$C_{i-1} = P_{i-1} + \frac{f_{i-1}}{2}(V_i - V_{i-1}) = P_{i-1} + \frac{f_{i-1}}{2}(V_i + V_{i+1})$$

$$C_i - C_{i-1} = P_i - P_{i-1} + \frac{f_1}{2}(V_{i+1} - V_i) - \frac{f_{i-1}}{2}(V_i + V_{i+1}).$$

Since $P_i - P_{i-1} = f_{i+1}V_i$ we have

$$C_i - C_{i-1} = \tfrac{1}{2}(2f_{i+1} - f_i - f_{i-1})V_i + \tfrac{1}{2}(f_i - f_{i-1})V_{i+1} = \frac{f_{i+1}}{2}V_i + \frac{f_{i-2}}{2}V_{i+1} .$$

$$|C_i - C_{i-1}|^2 = (C_i - C_{i-1})(C_i - C_{i-1}) = \tfrac{1}{4}(f_{i+1} + f_{i-2}) .$$

By Theorem 1 the triangle formed by the points C_i, C_{i-1}, and I is a right triangle.

$$d_i^2 + d_{i-1}^2 = |C_i - C_{i-1}|^2 = \tfrac{1}{4}(f_{i+1}^2 + f_{i-2}^2) .$$

We now proceed to find an explicit expression for the d_i which leads to the fact that the d_i form a recursive sequence.

Theorem 2.

$$d_i = \frac{f_{i+3} + f_{i-3}}{2\sqrt{10}} .$$

Proof. Let C_{i-2}, C_{i-1}, and C_i be three consecutive centers

$$d_i^2 + d_{i-1}^2 = \tfrac{1}{4}(f_{i+1}^2 + f_{i-2}^2)$$

$$d_{i-1}^2 + d_{i-2}^2 = \tfrac{1}{4}(f_i^2 + f_{i-3}^2)$$

$$d_i^2 - d_{i-2}^2 = \tfrac{1}{4}(f_{i+1}^2 - f_i^2 + f_{i-2}^2 - f_{i-3}^2) = \tfrac{1}{4}(f_{i+2}f_{i-1} + f_{i-4}f_{i-1}) \tag{7}$$

Also,

$$d_i^2 - d_{i-2}^2 = (d_i + d_{i-2})(d_i - d_{i-2}) = \frac{f_{i-1}\sqrt{10}}{2}(d_i - d_{i-2}) . \tag{8}$$

Combining (7) and (8) we have

$$d_i - d_{i-2} = \frac{1}{2\sqrt{10}}(f_{i+2} + f_{i-4})$$

and from Lemma 3

$$d_i + d_{i-2} = \frac{f_{i-1}\sqrt{10}}{2} .$$

Adding the last two equations we obtain

$$d_i = \frac{f_{i+2} + f_{i-4} + 10f_{i-1}}{4\sqrt{10}} .$$

It is a straightforward albeit tedious exercise to verify from Equation (1) that

$$f_{i+2} + f_{i-4} + 10f_{i-1} - 2f_{i+3} - 2f_{i-3} = 0$$

$$f_{i+2} + f_{i-4} + 10f_{i-1} = 2(f_{i+3} + f_{i-3})$$

$$\therefore d_i = \frac{f_{i+3} + f_{i-3}}{2\sqrt{10}}$$

Theorem 3.

$$d_{i+2} = d_{i+1} + d_i .$$

Proof.

$$d_{i+1} + d_i = \frac{1}{2\sqrt{10}}(f_{i+4} + f_{i-2} + f_{i+3} + f_{i-3})$$

$$= \frac{1}{2\sqrt{10}}(f_{i+5} + f_{i-1}) = d_{i+2}$$

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