THE GENERALIZED FIBONACCI NUMBER AND ITS RELATION TO WILSON'S THEOREM

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In this paper we consider the generalized Fibonacci second-order recurrence relation

(1) $$U_{k+2} = xU_{k+1} + yU_k ,$$

with x and y variables. Then for certain x and y in (1) we introduce the following new theorems:

Theorem 1. If $U_{p-1} \equiv 0 \pmod{p^2}$, then $p > 3$ is *always* an odd prime.

Corollary 1. If $U_p + 1 \equiv 0 \pmod p$ then $p > 3$ is *always* an odd prime.

Corollary 2. If $U_p + 1 \equiv 0 \pmod{p^2}$ or $\pmod{p^3}$ then $(p-1)! + 1 \equiv 0$ respectively $\pmod{p^2}$ or $\pmod{p^3}$.

In the Addenda of this paper we also prove: If

$$F_n = k_1F_{n-1} + k_2F_{n-2} ,$$

(where k_1 and k_2 are arbitrary constant numbers), then the following relation *always* holds

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^n k_2^n ,$$

where

$$F_0 = 1, \quad F_1 = k_1, \quad F_2 = k_1^2 + k_2, \cdots .$$

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For clarity we write (1) as

(2) $$U_k = x_kU_{k-1} + y_kU_{k-2} ,$$

where $k \geqslant 3$ is a positive integer, and the x_k, y_k are arbitrary variables.

$$U_k = x_kU_{k-1} + y_kU_{k-2}, \quad k \geqslant 2.$$

If $x_k = 2k - 1$ and $y_k = -(k-1)^2$, then (2) becomes

(3) $$U_{k+1} = (2k+1)U_k - k^2U_{k-1} .$$

What we want to show next is that if in addition to (3) we let

(3b) $$U_k = kU_{k-1} + (k-1)!,$$

then

$$U_{k+1} = (k+1)U_k + k!.$$

To see this,

$$\begin{aligned} U_{k+1} &= (2k+1)(kU_{k-1} + (k-1)!) - k^2U_{k-1} = 2k^2U_{k-1} + kU_{k-1} - k^2U_{k-1} + (2k+1)(k-1)! \\ &= k^2U_{k-1} + kU_{k-1} + 2k! + (k-1)! = (k+1)(kU_{k-1} + (k-1)!) + k! = (k+1)U_k + k!, \end{aligned}$$

which is (3b) with k replaced by $k+1$. The proof is complete by induction. We then conclude that Eq. (3) may be written in the following two ways:

(4) $$U_{k+1} = (2k+1)U_k - k^2U_{k-1} = (k+1)U_k + k!,$$

where $k \geq 2,\ U_1 = 1,\ U_2 = 3,\ U_3 = 11, \cdots$.

H. Gupta has noticed that the sequence $1, 3, 11, 50, \cdots, U_{k+1} = (k+1)U_k + k!$ is really the second column of the array of STIRLING NUMBERS OF THE FIRST KIND. See Riordan [4], pp. 33 and 48. Of course, in the table the signs are alternating.

From page 33 of [4] we find

(A) $$s(k+1, n) = s(k, n-1) - ks(k, n)$$

so that we note that if $n = 2$, we get

$$s(k+1, 2) = s(k, 1) - ks(k, 2)$$

and, from the table on page 48 of [4], we note

$$s(k, 1) = (-1)^{k+1}(k-1)!.$$

Now let

$$V_k(-1)^{k+1} = s(k+1, 2),$$

then (A) becomes

$$V_k(-1)^{k+1} = (-1)^{k+1}(k-1)! - kV_{k-1}(-1)^k$$

or equivalently

$$V_{k+1} = kV_k + k!$$

which agrees with (4) for $k+1$. Q.E.D.

It is of course evident that

(5) $$m(m-2)!/m! = 1/(m-1),$$

and also that

(6) $$U = 2!(1) + 1!$$

(by (4)). Then, since $U_3 = 3U_2 + 2!$, we combine this equation with (5, with m = 3) and (6), which leads to $U_3 = 3!(1 + 1/2) + 2!$, and in the exact way we get

(7) $$U_4 = 4!(1 + 1/2 + 1/3) + 3!.$$

Then in the exact way we derived (7), step-by-step (with added induction we prove that

(8) $$U_k = k!(1 + 1/2 + 1/3 + \cdots + 1/(k-1)) + (k-1) = k!\left(\sum_{r=1}^{k} 1/r\right),$$

for $k = 1, 2, 3, \cdots$. (It may be interesting to emphasize the fact that we have found the explicit formula

$$\sum_{r=1}^{k} 1/r = U_k/k!.)$$

Now, using the well known fact that

(9) $$\phi(k-1) = \sum_{r=1}^{k} 1/r \equiv 0 \pmod{k^2},$$

if and only if $k > 3$ is an odd prime (see 1), we are in a position to prove the following theorems:

(10) *Theorem 1.* If $U_{p-1} \equiv 0 \pmod{p^2}$, then $p > 3$ is always an odd prime. The proof is immediate by combining (8, with $k = p-1$) with (9) which leads to the congruence $U_{p-1} = (p-1)!\phi(p-1) \equiv 0 \pmod{p^2}$.

(10a) *Corollary 1.* If $U_p + 1 \equiv 0 \pmod p$, then $p > 3$ is always an odd prime.

The proof of Corollary 1 is immediate by combining (3b, with k replaced by some odd prime number $p > 3$) with Wilson's theorem (Wilson's theorem: $(p-1)! + 1 \equiv 0 \pmod p$, if and only if p is a prime number), since

(10b) $$U_p + 1 \equiv pU_{p-1} + (p-1)! + 1 \equiv 0 \pmod p.$$

(10c) *Corollary 2.* If $U_p + 1 \equiv 0 \pmod{p^2}$ or $\pmod{p^3}$, then $(p-1)! + 1 \equiv 0$ respectively $\pmod{p^2}$ or $\pmod{p^3}$.

We easily prove (10c) by combining (10b) with (10). Since this leads to

(10d) $$U_p + 1 \equiv (p-1)! + 1 \pmod{p^3}.$$

ADDENDA

1. We write the following familiar congruence (see 2):

(11) If $p > 3$ is a prime then $(p-1)! \equiv pB_{p-1} - p \pmod{p^2}$,

where B is a Bernoulli number. Now, combining (11) with (10d) we have

(12) $$U_p \equiv pB_{p-1} - p \pmod{p^2}.$$

(13) 2. N. Neilsen (see 3) proved that: If $p = 2n+1$, $P = 1\cdot 3\cdot 5 \cdots (2n-1)$, and $p > 3$ is a prime, then

$$P \equiv (-1)^n 2^{3n} n! \pmod{p^3}.$$

Now, combining (10d) with the results in (13) leads to

(14) $$U_{2n+1} \equiv (-1)^n 2^{4n} (n!)^2 \pmod{p^3}, \quad \text{where} \quad 2n+1 = p \text{ is a prime } > 3.$$

It is easy to prove that

(15) $$((k-1)!)^2 = U_k^2 + U_{k-1}U_k - U_{k-1}U_{k+1} = F(k-1).$$

Proof. In (3c) we have $U_k - kU_{k-1} = (k-1)!$, we then put $(U_k - kU_{k-1})^2 = F(k-1)$, and this leads to

(15a) $$U_{k+1} = (2k+1) - k^2 U_{k-1},$$

where, since (15a) is identical with (4), we have proved that (15) holds. Now, in (15) we let $n = k-1$, so that

$$(n!)^2 = U_{n+1}^2 + U_n U_{n+1} - U_n U_{n+2} = F(n),$$

and combining this identity with (14), we have:

(16) $$U_{2n+1} \equiv (-1)^n 2^{4n} (F(n)) \pmod{p^3},$$

where $2n+1 = p$ is a prime > 3.

3. A generalized version of (4) may be derived in the following way: Put

(17) $$U_k = U_{k-1}x_k + (k-1)!$$

(where the x are arbitrary variables). Then, multiplying (17) through by k, we have

(17a) $$kU_k = kU_{k-1}x_k + k!,$$

but in (17) it is evident that

(17b) $$U_{k+1} = U_k x_{k+1} + k!,$$

and subtracting (17a) from this equation we get

(18) $$U_{k+1} = (k + x_{k+1})U_k - kx_k U_{k-1}.$$

Example of 3. We easily prove (4) with (17b) and (18), if we let

$$x_k = k, \quad x_{k+1} = k+1, \cdots, x_{k+j} = k+j \quad (j = 0, 1, 2, \cdots).$$

4. In conclusion, it may be interesting to note: If

(19) $$F_n = k_1 F_{n-1} + k_2 F_{n-2},$$

(where k_1 and k_2 are arbitrary constants) then the following relation always holds:

(19a) $$F_n^2 - F_{n+1}F_{n-1} = (-1)^n k_2^n,$$

where $F_0 = 1, \quad F_1 = k_1, \quad F_2 = k_1^2 + k_2, \cdots$.

Proof. In (19) we may write $F_{n+1} = k_1 F_n + k_2 F_{n-1}$, and combining this with (19a), we have

(20) $$k_1 F_n F_{n-1} + k_2 F_{n-1}^2 = F_n^2 + (-1)^{n+1} k_2.$$

Now, we multiply both sides of (20) by k_2 and then add

$$k_1^2 F_n^2 + k_1 k_2 F_n F_{n-1}$$

to both sides of the result which leads to

$$k_1^2F_n^2 + 2k_1k_2F_nF_{n-1} + k_2^2F_{n-1}^2 = k_1^2F_n^2 + k_2F_n^2 + k_1k_2F_nF_{n-1} + (-1)^{n+1}k_2^{n+1} \quad (20a)$$

It is easily seen that

$$F_{n+2} = k_1F_{n+1} + k_2F_n = k_1^2F_n + k_1k_2F_{n-1} + k_2F_n ,$$

and combining this equation with (20a), we have

$$(k_1F_n + k_2F_{n-1})^2 = F_{n+1}^2 = F_{n+2}F_n + (-1)^{n+1}k_2^{n+1} . \quad (20b)$$

In the same way we found (20b), we proceed step-by-step (with added induction) and prove that the identities in (19) and (19a) are correct.

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PYTHAGOREAN TRIANGLES

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ABSTRACT

The first section of "Pythagorean Triangles" is primarily a portion of the history of pythagorean triangles and related problems. However, some new results and some new proofs of old results are presented in this section. For example, Fermat's Theorem is used to prove:

Levy's Theorem. If (x,y,z) is a pythagorean triangle such that $(7,x) = (7,y) = 1$, then 7 divides $x + y$ or $x - y$.

The historical discussion makes it reasonable to define pseudo-Sierpinski triangles as primitive pythagorean triangles with the property that $x = z - 1$, where z is the hypotenuse and x is the even leg. Whether the set of pseudo-Sierpinski triangles is finite or infinite is an open question. Some elementary, but new, results are presented in the discussion of this question.

An instructor of a course in Number Theory could use the material in the second section to present a coherent study of Fermat's Last Theorem and Fermat's method of infinite descent. These two results are used to prove the following familiar results.

(1A) *No pythagorean triangle has an area which is a perfect square.*

(2A) *No pythagorean triangle has both legs simultaneously equal to perfect squares.*

(3A) *It is impossible that any combination of two or more sides of a pythagorean triangle be simultaneously perfect squares.*

If 2 is viewed as a natural number for which Fermat's Last Theorem is true, then the following are obvious generalizations of 1A, 2A, and 3A.

(1B) *If* k *is an integer for which Fermat's Last Theorem holds, then there is no primitive pythagorean triangle whose area is a* k^{th} *power of some integer.*

(2B) *If* k *is some integer for which Fermat's Last Theorem is true, then there is no pythagorean triangle with the legs both equal to* k^{th} *powers of natural numbers.*

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