Now if \( x \equiv 0 \pmod{p} \), \( F_x(n) \equiv 1 \pmod{p} \) for all \( n \), by the definition of \( F_x(n) \).

If \( x \not\equiv 0 \pmod{p} \), from Lemma 5 there exists a number \( \alpha \) such that \( F_x(\alpha) \equiv 0 \pmod{p} \), we assume that \( \alpha \) is the least such number, and \( \alpha > 1 \) since \( F_x(1) = 1 \) for all \( x \). It can be shown inductively that \( F_x(n+\alpha) = sF_x(n) \pmod{p} \) for all \( n \), where \( s = F_x(\alpha + 1) \pmod{p} \), and \( s \neq 0 \) since \( s \equiv 0 \) would imply \( F_x(\alpha - 1) \equiv 0 \pmod{p} \). Then if \( F_x(r) \equiv 0 \pmod{p} \), there exists \( r' \) such that

\[
r' = r \pmod{\alpha}, \quad 0 < r' < \alpha, \quad \text{and} \quad F_x(r') \equiv 0 \pmod{p}.
\]

By the definition of \( \alpha \), \( r' < \alpha \) is absurd, therefore \( r' = \alpha \).

Let \( P \) be prime and \( p \) a prime factor of \( F_x(P). \) Then

\[
F_x(P) \equiv 0 \pmod{p} \quad \text{and} \quad x \not\equiv 0 \pmod{p}
\]

Thus \( P = 0 \pmod{\alpha} \) and since \( P \) is prime, \( P = \alpha \). Let \( p' \) be either \( p, p - 1, \) or \( p + 1 \), such that

\[
F_x(p') \equiv 0 \pmod{p}
\]

(from Lemma 3). Then \( p' \) is an integral multiple of \( P \) and the theorem follows.

I mentioned this result to Dr. P.M. Lee of York University and he has pointed out to me that Lemma 3 can be derived from H. Siebeck's work on recurring series (L.E. Dickson, History of the Theory of Numbers, p. 394f). A colleague of his has also discovered a non-elementary proof of the above theorem.

I am myself only an amateur mathematician, so I would ask you to excuse any resulting awkwardnesses in my presentation of this theorem and proof.

Yours faithfully,
Alexander G. Abercrombie

[Continued from Page 146.]

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There is room for considerable work regarding possible lengths of periods. For various values of \( p \) and \( q \) we found periods of lengths: 1, 2, 8, 9, 17, 25, 33, 35, 42, 43, 61, 69.

**GENERALIZED PERIODS**

For various sequence types, it is possible to arrive at generalized periods. Some examples are the following.

\((p, p - 1): 2p - 2, 2p - 3, 2p - 3, 2p - 2, 2p, 2p + 2, 2p + 3, 2p + 2, 2p, \) where \( p \) is large enough to make all quantities positive.

\((p, p): 2p, 2p + 2, 2p + 1, 2p - 1, 2p, 2p - 1, 2p + 1, \) where \( p > 2 \), and many others.

\((p + 1, p): 2p - 1, 2p, 2p + 2, 2p + 4, 2p + 5, 2p + 4, 2p + 2, 2p - 1 \) for \( p > 3 \) (Period of length 9)

\(2p, 2p + 1, 2p + 5, 2p + 5, 2p + 5, 2p + 1, 2p - 3, 2p - 1, 2p - 1, 2p + 4, 2p + 4, 2p + 7, 2p + 3, 2p + 2, 2p - 3, 2p - 3, 2p + 2, 2p + 3, 2p + 8, 2p + 7, 2p + 4, 2p + 4, 2p - 1, 2p - 1, 2p - 3, \) for \( p > 24 \) (Period of length 26), and many others.

A schematic method was used which made the work of arriving at these results somewhat less laborious.

**NON-PERIODIC SEQUENCES**

In studying the sequences \((3,4),\) non-periodic sequences of a quasi-periodic type were found. They have the peculiar property that alternate terms form a regular pattern in groups of four, while the intermediate terms between these pattern terms become unbounded. This situation arises in sequences \((p, q)\) for which \( q \) is greater than \( p \).

As an example of such a non-periodic sequence in the case \((4, 7)\) the sequence beginning with 1,3,4, follows:

1, 3, 4, 37, 59, 124, 25, 17, 2, 6, 3, 27, 22, 93, 20, 34, 3, 13, 3, 35, 13, 99, 14, 58, 4, 31, 3, 58, 9, 148, 12, 121, 4, 72, 3, 129, 8, 312, 11, 279, 4, 178, 3, 317, 8, 751, 10, 663, 4, 466, 3, 819, 8, 1922, 10, 1687, 4, 1183, 3, 2074, 8, 4850, 10, 4249, 4, 2976, 3, 5211, 8, 12170, 10, ...

Note the regular periodicity of 3,8,10,4 with the sets of intermediate terms increasing as the sequence progresses.

The various types of non-periodic sequence for \((4, 7)\) are:

[Continued on Page 184.]