ELEMENTARY PROBLEMS AND SOLUTIONS

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DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-304 Proposed by Sidney Kravitz, Dover, New Jersey.

According to W. Hope-Jones, "The Bee and the Pentagon," *The Mathematical Gazette,* Vol. X, No. 150, 1921 (Reprinted Vol. LV, No. 392, March 1971, Page 220), the female bee has two parents but the male bee has a mother only. Prove that if we go back n generations for a female bee she will have F_n male ancestors in that generation and F_{n+1} female ancestors, making a total of F_{n+2} ancestors.

B-305 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n} = L_{2n} \sum_{k=1}^{n} L_{2n+4k-2} .$$

B-306 Proposed by Frank Higgins, North Central College, Naperville, Illinois.

Prove that

$$F_{8n+1} - 1 = L_{2n} \sum_{k=1}^{n} L_{2n+4k-1} .$$

B-307 Proposed by Verner E. Hoggatt, Jr., California State University, San Jose, California.

Let

$$(1 + x + x^2)^n = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots,$$

(where, of course, $a_{n,k} = 0$ for $k > 2n$). Also let

$$A_n = \sum_{j=0}^{\infty} a_{n,4j}, \quad B_n = \sum_{j=0}^{\infty} a_{n,4j+1}, \quad C_n = \sum_{j=0}^{\infty} a_{n,4j+2}, \quad D_n = \sum_{j=0}^{\infty} a_{n,4j+3} .$$

Find and prove the relationship of A_n, B_n, C_n, and D_n to each other. In particular, show the relationships among these four sums for $n = 333$.

B-308 Proposed by Phil Mana, Albuquerque, New Mexico.

(a) Let $c_n = \cos(n\theta)$ and find the integers a and b such that $c_n = ac_{n-1} + bc_{n-2}$ for $n = 2, 3, \cdots$.

(b) Let r be a real number such that $\cos(r\pi) = p/q$, with p and q relatively prime positive integers and q not in 1, 2, 4, 8, ⋯ . Prove that r is not rational.

B-309 Corrected Version of B-284.

Let $z^2 = xz + y$ and let k, m, and n be nonnegative integers. Prove that:

(a) $z^n = p_n(x,y)z + Q_n(x,y)$, where p_n and q_n are polynomials in x and y with integer coefficients and p_n has degree $n-1$ in x for $n > 0$.

(b) There are polynomials r, s, and t, not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0 .$$

SOLUTIONS

THE EDITOR'S DIGITS

B-280 Proposed by Maxey Brooke, Sweeney, Texas.

Identify $A, E, G, H, J, N, O, R, T, V$ as the ten distinct digits such that the following holds with the dots denoting some seven-digit number and ϕ representing zero:

$$\begin{array}{r} VERNER \\ \times\underline{\qquad\quad E} \\ \cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ -\underline{R\phi\phi\phi\phi JR} \\ HOGGATT \end{array}$$

Solution by Paul S. Bruckman, University of Illinois, Chicago Circle Campus.

The unique solution to the problem is the following:

$$\begin{array}{r} 971471 \\ \times\underline{\qquad 7} \\ 6800297 \\ -\underline{1000031} \\ 5800266 \end{array}$$

i.e., we have:

$$\begin{array}{c} AEGHJNORTV \\ 2705348169 \end{array}$$

Proof. Let the product $VERNER \times E$ be denoted by P in this discussion, and let the first digit of P be denoted by Y. Since P is a 7-digit number, and $VERNER$ is a 6-digit number, then $E \geqslant 2$. Since R and H are both at least 1, their total must be at least 3 (since $R \neq H$); hence, $E \geqslant 4$ and $Y \geqslant 3$.

Since $R + T \equiv ER \pmod{10}$, we initially obtain 39 possibilities for E, T, R with $E \geqslant 4$. Taking into account the possible values of J, we are left with 26 possibilities for E, T, R, J.

Now $Y \leqslant E - 1$ (since $V \leqslant 9$); moreover, since $H \geqslant 1$, we must have $R \leqslant E - 2$. Taking this requirement into account, we further reduce the list to only 13 possibilities. By a slightly tedious but manageable process of elimination, we conclude the result indicated above.

Also solved by John W. Milsom, C. B. A. Peck, Richard D. Plotz, and the Proposer.

ONES FOR TEE

B-281 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $T_n = n(n+1)/2$. Find a positive integer b such that for all positive integers m, $T_{11\cdots1} = 11\cdots1$, where the subscript on the left side has m 1's as the digits in base b and the right side has m 1's as the digits in base b^2.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

More will be shown to be true. Suppose the base on the right side is the positive integer c, instead of b^2. The equality for $m = 1$ is automatically satisfied and for $m = 2$ is $(1+b)(2+b) = 2(1+c)$, i.e., $3b + b^2 = 2c$. For $m = 3$ the resulting equation is

$$(1+b+b^2)(2+b+b^2) = 2(1+c+c^2).$$

These last two equations in b and c force $b^2 = 2b+3$ and hence $b = 3$ (since it is a positive integer), and $c = b^2 = 9$. Finally as $(3^m - 1)(3^m + 1) = (3^{2m} - 1)$ then $T_{11\cdots 1}$, in base 3, equals $11 \cdots 1$, in base 9, for all positive integers greater than 2.

Also solved by Paul S. Bruckman, Herta T. Freitag, C.B.A. Peck, Bob Prielipp, Paul Smith, Gregory Wulczyn, and the Proposer.

LUCAS RIGHT TRIANGLES

B-282 Proposed by Herta T. Freitag, Roanoke, Virginia.

Characterize geometrically the triangles that have

$$L_{n+2}L_{n-1}, \quad 2L_{n+1}L_n, \quad \text{and} \quad 2L_{2n} + L_{2n+1}$$

as the lengths of the three sides.

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

Since

$$[2L_{2n} + L_{2n+1}]^2 = [L_{2n} + L_{2n+2}]^2 = [L_{n-1}L_{n+1} + 3(-1)^n + L_nL_{n+2} + 3(-1)^{n+1}]^2$$

(see the Solution to Problem B-256, p. 221, *The Fibonacci Quarterly,* April 1974)

$$= [L_{n-1}L_{n+1} + L_nL_{n+2}]^2 = [(L_{n-1} + L_n)L_{n+1} + L_n^2]^2 = [L_{n+1}^2 + L_n^2]^2 = [2L_{n+1}L_n]^2 + [L_{n+1}^2 - L_n^2]^2$$

$$= [2L_{n+1}L_n]^2 + [(L_{n+1} + L_n)(L_{n+1} - L_n)]^2 = [2L_{n+1}L_n]^2 + [L_{n+2}L_{n-1}]^2,$$

the triangles are right triangles.

Also solved by Richard Blazej, Paul S. Bruckman, Wray G. Brady, C.B.A. Peck, Gregory Wulczyn, and the Proposer.

RATIONAL APPROXIMATION OF COS $\pi/6$ AND SIN $\pi/6$

B-283 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the ordered triple (a, b, c) of positive integers with $a^2 + b^2 = c^2$, a odd, $c < 1000$, and c/a as close to 2 as possible. [This approximates the sides of a 30°, 60°, 90° triangle with a Pythagorean triple.]

Solution by Paul Smith, University of Victoria, Victoria, B.C., Canada.

It is clearly sufficient to find a triple of the form $(u^2 - v^2, 2uv, u^2 + v^2)$, with u, v of opposite parity. We must then find the minimum value for $u^2 + v^2 < 1000$ of

$$\left|2 - \frac{u^2+v^2}{u^2-v^2}\right| = \left|\frac{u^2-3v^2}{u^2-v^2}\right| .$$

If $\left|u^2 - 3v^2\right| = 2$ then u, v are of the same parity and a is even. Hence, if $\left|u^2 - 3v^2\right| > 1$,

$$\left|\frac{u^2-3v^2}{u^2-v^2}\right| > \left|\frac{u^2-3v^2}{u^2+v^2}\right| \geq \frac{3}{1000} .$$

For $u^2 + v^2 < 1000$ the Pellian equation $|u^2 - 3v^2| = 1$ has solutions $(u,v) = (2,1), (7,4), (26, 15)$. The solution $(26, 15)$ yields the triple $(451, 780, 901)$ which is best possible, since

$$\left|2 - \frac{901}{451}\right| = \frac{1}{451} < \frac{3}{1000} .$$

Also solved by Paul S. Bruckman, Gregory Wulczyn, and the Proposer.

CORRECTED AND REINSERTED

Problem B-284 has been corrected and reinserted as B-309 above.

VERY SLIGHT VARIATION ON A PREVIOUS PROBLEM

B-285 Proposed by Barry Wolk, University of Manitoba, Winnipeg, Manitoba, Canada.

Show that

$$F_{k(n+1)}/F_k = \sum_{k=0}^{[n/2]} (-1)^{r(k-1)} \binom{n-r}{r} L_k^{n-2r} .$$

Solution by C.B.A. Peck, State College, Pennsylvania.

This was H-135, Part II and was proved by induction on n in *The Fibonacci Quarterly,* Vol. 7, No. 5, p. 519. (The exponent of -1 in that problem has + instead of $-$, but $(-1)^{2r} = 1$.)

Also solved by P.S. Bruckman and the Proposer.

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