FORMAL PROOF OF EQUIVALENCE OF TWO SOLUTIONS OF THE GENERAL PASCAL RECURRENCE

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There have been numerous studies of the general Pascal recurrence relation

(1) $$f(x+1,\,y+1) - f(x,y+1) - f(x,y) = 0.$$

Defining

$$\Delta_x f(x,y) = f(x+1,\,y) - f(x,y), \qquad \Delta_y f(x,y) = f(x,\,y+1) - f(x,y),$$
$$E_x f(x,y) = f(x+1,\,y), \qquad E_y f(x,y) = f(x,\,y+1),$$

Milne-Thomson [8] notes that Eq. (1) may be recast in the form of the partial difference equation with constant coefficients

(2) $$E_y \Delta_x f(x,y) - f(x,y) = 0$$

for which one may write down the formal solution

(3) $$f(x,y) = (1 + E_y^{-1})^x \phi(y),$$

where $\phi(y)$ is an arbitrary function. Hence Milne-Thomson finds the classical formal solution (finite series when x is a positive integer)

(4) $$f(x,y) = \sum_{k=0}^{\infty} \binom{x}{k} \phi(y-k).$$

There is then also an alternative way to write such a formal series solution:

(5) $$f(x,y) = \sum_{k=0}^{\infty} \binom{x}{k} \phi(y-x+k).$$

These are old and well-known results, easily found in other treatises on the calculus of finite differences. The method of generating functions is used in [8] also and the results agree with the two possible series solutions we have quoted above.

As for getting a nice, elegant, explicit formula for the general solution to such partial difference equations (and of higher order), we would be remiss if we did not mention the two valuable papers of Carlitz [3] and [4]. Anyone working with arrays of numbers ought to consult these papers for a close-hand study of the interesting way Carlitz handles the equations. These papers deal with formulas for sums of powers of the natural numbers and the formulas involve Bernoulli and Stirling numbers as well as expansions of differential operators.

Most recently, Eq. (1) has arisen in some interesting new work on partitions [1], [5]. Carlitz's solution of a recurrence in [5] has now attracted Hansraj Gupta [7] who has announced the following result:

Theorem. Let $c(n+1,\,k) = c(n,k) + c(n,\,k-1)$, with $c(n,0) = a(n)$, $c(1,k) = b(k)$, $n,k \geqslant 1$, where $a(n)$ and $b(k)$ are arbitrary functions of n and k, respectively. Then, explicitly,

(6) $$c(n,k) = \sum_{r=k}^{n-1} \binom{r-1}{k-1} a(n-r) + \sum_{r=0}^{k-1} \binom{n-1}{r} b(k-r), \qquad k \geqslant 1.$$

This generalizes the solutions and formulas given in [1] and [5]. What we propose to do here is to show the equivalence of Gupta's formula (6) and the well-known formal series solution (4). We show that the one implies the other. A simple combinatorial identity listed in [6] equivalent to the Vandermonde convolution (addition formula) is used in the discussion.

We first need to reformulate Gupta's result in the notation of the present paper. In our notation, formula (6) becomes

(7) $$f(x,y) = \sum_{r=y}^{x} \binom{r-1}{y-1} f(x-r,\,0) + \sum_{r=0}^{y-1} \binom{x}{r} f(0,\,y-r),$$

for integers $x,y \geqslant 1$.

In the steps below we need at one spot formula (3.4) from [6]:

$$\sum_{r=\alpha}^{\beta-\gamma} \binom{r}{\alpha}\binom{\beta-r}{\gamma} = \binom{\beta+1}{\alpha+\gamma+1} \tag{8}$$

We find then that assuming (4)

$$\sum_{r=y}^{x} \binom{r-1}{y-1} f(x-r,0) = \sum_{r=y}^{x} \binom{r-1}{y-1} \sum_{j=0}^{\infty} \binom{x-r}{j} \phi(-j) = \sum_{j=0}^{\infty} \phi(-j) \sum_{r=y}^{x} \binom{r-1}{y-1}\binom{x-r}{j}$$

$$= \sum_{j=0}^{\infty} \phi(-j) \sum_{r=y-1}^{x-1} \binom{r}{y-1}\binom{x-1-r}{j} = \sum_{j=0}^{\infty} \phi(-j) \sum_{r=y-1}^{x-1-j} \binom{r}{y-1}\binom{x-1-r}{j}$$

$$= \sum_{j=0}^{\infty} \phi(-j) \binom{x}{y+j} = \sum_{r=y}^{\infty} \binom{x}{r} \phi(y-r),$$

so that we have shown in fact

$$\sum_{r=y}^{x} \binom{r-1}{y-1} f(x-r,0) = \sum_{r=y}^{\infty} \binom{x}{r} \phi(y-r). \tag{9}$$

Upon adding the trivial relation (clear from (4))

$$\sum_{r=0}^{y-1} \binom{x}{r} f(0,y-r) = \sum_{r=0}^{y-1} \binom{x}{r} \phi(y-r)$$

to both sides of (9), we find that we have proved (7). Conversely, it is easy to see how to follow the steps in reverse so that series (4) can be broken into two parts as specified in (7). Solving (1) in terms of an arbitrary function ϕ is equivalent to setting up the two sequences $f(x-r,0)$ and $f(0,y-r)$. We leave aside the discussion of convergence questions.

As a final observation, Cadogan [2] has shown how to solve the slight extension of (1): $f(k,n) = pf(k,n-1) + qf(k-1,n-1)$, where p,q are arbitrary fixed constants. He interprets the resulting arrays in terms of arithmetic and geometric sequences for certain choices of parameters. There is nothing new in this, but his paper is a worthwhile pedagogical survey written at an elementary level. Similarly, there is nothing "new" in the present paper, but we have spelled out the manipulations of our proof to show how one actually does the verification of equivalence. In a similar way the reader may write out the same argument using (5) instead of (4). Of course, the equivalence of these with Gupta's (6) has been shown here only when x,y are integers, and the reader must bear in mind that (4) and (5) are more general than (6) because they hold in cases where x,y are not integers. The series manipulations leading to (9) are easily justified because the series are really finite series, $\binom{x}{r} = 0$ for example, when $r > x$, x being a non-negative integer.

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