APPROXIMATION OF IRRATIONALS WITH FAREY FIBONACCI FRACTIONS

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The author in [1] had defined the Farey sequence of Fibonacci Numbers as follows:

A Farey sequence of Fibonacci Numbers of order f_n is the set of all possible fractions f_i/f_j $j \leqslant n$ put in ascending order of magnitude $\left\{ n, i, j \geqslant 0 \right\}$, are positive integers; f_n denotes the n^{th} Fibonacci Number, $0/f_{n-1}$ is the first fraction. This set is denoted by $f \cdot f_n$.

We also defined an "INTERVAL" in $f \cdot f_n$ to consist of all fractions in $f \cdot f_n$ between fractions of the form

$$\left(\frac{1}{f_i}, \frac{1}{f_{i-1}} \right) \quad i \leqslant n \quad (f_{j-1}, f_j) \quad j > 0 \, ,$$

j a positive integer. Two symmetry properties were established:

(1) Let $h/k, h'/k', h''/k''$ be three consecutive fractions in $f \cdot f_n$ all greater than 1. Further let $f_{i-1} < h/k < h'/k' < h''/k'' < f_i$. Then

(a)
$$\frac{h+h''}{k+k''} = \frac{h'}{k'} \, ,$$

(b)
$$kh' - hk' = f_{i-2} \, .$$

(1*) Let h/k, h'/k', h''/k'' be three consecutive fractions in $f \cdot f_n$ all less than 1. Further let $1/f_i < h/k < h'/k' < h''/k'' < 1/f_{i-1}$. Then

(a)
$$\frac{h+h''}{k+k''} = \frac{h'}{k'}$$

(b)
$$kh' - hk' = f_{i-2} \, .$$

Many other relations of symmetries besides these are proved in [1]. 1(a), 1(b) are similar to properties which are preserved by the Farey Sequence also. Actually instead of arranging Fibonacci fractions, in ascending order, we had arranged fractions of the sequence $U_n = U_{n-1} + U_{n-2}$ $U_1 > U_0 > 0$ integers, still some of the properties will remain. However, with the Fibonacci Sequence we get more symmetries.

The problem we discuss in this paper is the approximation of irrationals with Farey Fibonacci Fractions. We prove some theorems on best approximations.

Definition. Consider any $f \cdot f_n$. Form a new ordered set $ff_{n,1}$ consisting of all rationals in $f \cdot f_n$, together with mediants of consecutive rationals in $f \cdot f_n$. Define recursively $f \cdot f_{n,r+1}$ as all the rationals in $f \cdot f_{n,r}$ together with mediants of consecutive rationals in $f \cdot f_{n,r}$. The first rational in $f \cdot f_{n,r+1}$ is rewritten as $0/f_{n+r}$. We now define

$$F \cdot F_n = \bigcup_{r=1}^{\infty} f \cdot f_{n,r} \, .$$

Propositions. $F \cdot F_n$ is dense in $(0, \infty)$ in the sense that its closure gives the interval $(0, \infty)$. This implies that every irrational "θ" can be approximated by a sequence of rationals h/k in $F \cdot F_n$. Without loss of generality we consider only the case $\theta > 0$, for $\theta < 0$ can be approximated by $-h/k$ where h/k belong to $F \cdot F_n$. They are all quite obvious, and can be easily seen from (1) and (1*).

We now begin with a theorem on best approximation.

Theorem 1. (a) Let θ be an irrational > 1, say $f_{i-1} < \theta < f_i$. Then there exist infinitely many rationals $h/k \in F{\cdot}F_n$ for each "n" such that

$$\left|\theta - \frac{h}{k}\right| < \frac{f_{i-2}}{\sqrt{5}k^2}\ .$$

(b) Let θ be an irrational < 1, say $1/f_i < \theta < 1/f_{i-1}$. Then there exist infinitely many $h/k \in F{\cdot}F_n$ for every "n" such that

$$\left|\theta - \frac{h}{k}\right| < \frac{f_{i-2}}{\sqrt{5}k^2}\ .$$

Moreover the constant $\sqrt{5}$ is the best possible in the sense that the assertion fails if $\sqrt{5}$ is replaced by a bigger constant.

Proof. We prove only Theorem 1(a). The proof of 1(b) is similar. In proving the theorem we follow the proof of Hurwitz theorem as given in Niven's book [2].

We need the well known lemma

Lemma. It is impossible to find integers x,y such that the two inequalities simultaneously hold.

$$\frac{1}{xy} \geqslant \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{y^2}\right); \qquad \frac{1}{x(x+y)} \geqslant \frac{1}{\sqrt{5}}\left(\frac{1}{x^2}+\frac{1}{(x+y)^2}\right)\ .$$

We don't give the proof of the lemma as it is known.

Now let "θ" lie between two consecutive fractions of $f{\cdot}f_{n,r}$, i.e., $a/b < \theta < c/d$. It is clear that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \qquad \text{and} \qquad \frac{a+c}{b+d} \in f{\cdot}f_{n,r+1}\ .$$

Now we shall show that at least one of these fractions, say h/k, satisfies

$$\left|\theta - \frac{h}{k}\right| < \frac{f_{i-2}}{\sqrt{5}k^2}\ .$$

Case 1. Let

$$\frac{a}{b} < \frac{a+c}{b+d} < \theta < \frac{c}{d},$$

and let

$$\theta - \frac{a}{b} \geqslant \frac{f_{i-2}}{\sqrt{5}b^2}; \qquad \theta - \frac{a+c}{b+d} \geqslant \frac{f_{i-2}}{\sqrt{5}(b+d)^2}; \qquad \frac{c}{d} - \theta \geqslant \frac{f_{i-2}}{\sqrt{5}d^2}$$

These three inequalities give rise to

$$\frac{c}{d} - \frac{a}{b} \geqslant \frac{f_{i-2}}{\sqrt{5}}\left(\frac{1}{b^2}+\frac{1}{d^2}\right) \qquad \text{and} \qquad \frac{c}{d} - \frac{a+c}{b+d} \geqslant \frac{f_{i-2}}{\sqrt{5}}\left(\frac{1}{d^2}+\frac{1}{(b+d)^2}\right)\ .$$

Now by properties 1(a) and 1(b) we get

$$\frac{1}{bd} \geqslant \frac{1}{\sqrt{5}}\left(\frac{1}{b^2}+\frac{1}{d^2}\right); \qquad \frac{1}{d(b+d)} \geqslant \frac{1}{\sqrt{5}}\left(\frac{1}{d^2}+\frac{1}{(b+d)^2}\right)$$

which is a contradiction according to the lemma.

Proceed similarly for the case

$$\frac{a}{b} < \theta < \frac{a+c}{b+d} < \frac{c}{d}.$$

Hence at least one of the three fractions, say h/k, gives

$$\left|\theta - \frac{h}{k}\right| < \frac{f_{i-2}}{\sqrt{5}k^2}\ .$$

One can very esasily see that there are infinitely many of them in $F \cdot F_n$ from the Propositions given and from the very definition of $F \cdot F_n$.

It is easy to see that $\sqrt{5}$ is the best possible constant. Consider the case when

$$\theta = \frac{1+\sqrt{5}}{2} .$$

Now $f_{i-2} = 1$, and so we have

$$\left| \theta - \frac{h}{k} \right| < \frac{1}{\sqrt{5}k^2}$$

for infinitely many h/k in $F \cdot F_n$. We can't obviously improve $\sqrt{5}$. This follows from the classical theorem of Hurwitz [2].

Note. (A) The counter example $(1+\sqrt{5})/2$ which Hurwitz gave is actually the

$$\lim_{n \to \infty} \frac{f_{n+1}}{f_n} .$$

(B) In the interval $(\frac{1}{3}, 3)$ $F \cdot F_n$ provides the same approximation as do the Farey Fractions for $f_{i-2} = 1$.

In Theorem 1 the constant $\sqrt{5}$ was seen as the best possible over the interval $(0, \infty)$. That is if $\sqrt{5}$ were replaced by a larger constant the theorems do not hold for all irrationals "θ" > 0. Now our question is the following: Is $\sqrt{5}$ the best possible constant for every "INTERVAL" (f_{i-1}, f_i)? The answer is in the affirmative in the sense that if $\sqrt{5}$ is replaced by a larger constant the theorem fails to hold in all "INTERVALS"

$$(f_{i-1}, f_i) \qquad \text{and} \qquad \left(\frac{1}{f_i}, \frac{1}{f_{i-1}} \right), \qquad i = 2, \cdots, \infty .$$

We now state and prove our final pair of theorems which are much stronger than Theorem 1.

Theorem 2a. Consider any "INTERVAL" $\{ f_{i-1}, f_i \}$. Let θ be an irrational which belongs to this interval. Then for any "n" there exists infinitely many

$$\frac{h}{k} \in F \cdot F_n$$

such that

$$\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2} .$$

The constant $\sqrt{5}$ is the best possible in the sense that if $\sqrt{5}$ are replaced by a larger constant the assertion fails for each "n" for some θ belonging to this interval.

Proof. The existence has already been established in Theorem 1. We shall concentrate on the bound

$$\frac{f_{i-2}}{\sqrt{5}} .$$

If we show that

$$\frac{f_{i-2}}{\sqrt{5}}$$

is the best possible constant when $n = 1$, it proves the theorem for using properties 1a, 1*a one can show $F \cdot F^*_{n+1} \subset F \cdot F_n$, where

$$F \cdot F^*_n = \{ x \in F \cdot F_n | x \geqslant 1 \}$$

Consider the interval

$$\left(\frac{f_{i-1}}{1}, \frac{f_i}{1} \right)$$

and call "s" the set

$$\left(\frac{f_{i-1}}{1}, \frac{f_i}{1} \right) .$$

Let

$$s_1 = \left(\frac{f_{i-1}}{1}, \frac{f_{i-1}+f_i}{1\,1}, \frac{f_i}{1} \right).$$

Defined recursively let s_{r+1} consist of all fractions in s_r, together with the mediants of consecutive fractions in s_r.

Let

$$S = \bigcup_{r=1}^{\infty} s_r .$$

Similarly let

$$s' = \left(\frac{1}{1}, \frac{2}{1} \right) \qquad \text{and} \qquad s'_1 = \left(\frac{1}{1}, \frac{1+2}{1+1}, \frac{2}{1} \right).$$

Define s'_{r+1} as all fractions in s'_r together with mediants of consecutive fractions in s'_r. Now let

$$S' = \bigcup_{r=1}^{\infty} s'_r \ .$$

What we are interested here is S and not $F \cdot F_n$. If we compare the sets s_r, and s'_r, the following can easily be seen.

(i) A one-one onto map can be established between s_r and s'_r as follows.

Map

$$\frac{\nu 1 + \mu \cdot 2}{\nu \cdot 1 + \mu \cdot 1} \rightarrow \frac{\nu f_{i-1} + \mu f_i}{\nu \cdot 1 + \mu \cdot 1} .$$

We call two such numbers corresponding numbers.

(ii) The map says that to every $(p/q) \in s_r$ there exists a unique $(p'/q) \in s'_r$ and conversely

(iii) The distance between the consecutive numbers in s_r is f_{i-2} times the distance between consecutive numbers in s'_r.

Now let

$$\theta_0 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \theta_1 = \frac{1+\sqrt{5}}{2} - 1.$$

Clearly

$$f_{i-1} + f_{i-2}\theta_1 = \theta'$$

is in s_r. Now if there exist infinitely many h_r/k_r in S with

$$\left| \theta' - \frac{h_r}{k_r} \right| < \frac{f_{i-2}}{ak_r^2}, \qquad a > \sqrt{5} \qquad r = 1, 2, \cdots$$

that (i), (ii), (iii) would imply that there exists infinitely many corresponding numbers h'_r/k_r in S' with

$$\left| \theta' - \frac{h'_r}{k_r} \right| < \frac{1}{ak_r^2} \qquad \text{with} \qquad a > \sqrt{5},$$

which is a contradiction according to Hurwitz theorem. Hence the theorem fails for θ' if $\sqrt{5}$ is replaced by a bigger constant.

Theorem 2b. Consider any interval

$$\left(\frac{1}{f_i}, \frac{1}{f_{i-1}} \right) .$$

Let

$$\theta \in \left(\frac{1}{f_i}, \frac{1}{f_{i-1}} \right)$$

be an irrational. Then there exists infinitely many h/k in $F \cdot F_n$ for all $n \geq i$ such that

$$\left| \theta - \frac{h}{k} \right| < \frac{f_{i-2}}{\sqrt{5}k^2} .$$

The constant here again in the best possible in the same sense as Theorem 2a.

Proof. The existence is already known. We just prove the converse for $F \cdot F_i$. It automatically follows for the other cases. Now let

$$s'' = \left(\frac{1}{f_i}, \frac{1}{f_{i-1}}\right) .$$

Let

$$s_1'' = \left(\frac{1}{f_i}, \frac{1+1}{f_{i-1}+f_i}, \frac{1}{f_{i-1}} \right).$$

Define recursively s_{r+1}'' as all fractions in s_r'' together with mediants of consecutive fractions in s_r''. Now

$$S'' = \bigcup_{r=1}^{\infty} s_r'' .$$

Clearly a one-one onto map exists between S'' and S.

Map

$$\frac{h}{k} \to \frac{k}{h} .$$

Consider the irrational $1/\theta' = \theta''$. Let there exist infinately many

$$\frac{h}{k} \in F \cdot F_i$$

with

$$\left|\theta'' - \frac{h_r}{k_r}\right| < \frac{f_{i-2}}{ak_r^2} .$$

Now if

$$\theta'' = \frac{h_r}{k_r} + \frac{\delta f_{i-2}}{ak_r^2}$$

then $|\delta| < 1/a$, Now this gives that

$$\left|\theta' - \frac{k_r}{h_r}\right| < \frac{f_{i-2}}{a\left(h_r + \dfrac{\delta f_{i-2}}{k_r}\right)(h_r)}$$

for infinitely many k_r/h_r in S. Now choose any $\beta > \delta$ with $\sqrt{5} < \beta < a$. Then we get

$$\left|\theta' - \frac{k_r}{h_r}\right| < \frac{f_{i-2}}{\beta h_r^2}$$

for infinitely many k_r/h_r in S, i.e., for all $r > r_0$ $(\beta > \sqrt{5})$.

This is a contradiction according to Theorem 2a and so Theorem 2b is proved.

REFERENCES

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