FIBONACCI MULTI-MULTIGRADES

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Readers of *The Fibonacci Quarterly* will probably be familiar with multigrades. Here are two examples:

(1) $$1^m + 6^m + 8^m = 2^m + 4^m + 9^m \qquad (m = 1, 2)$$

and

(2) $$1^m + 5^m + 8^m + 12^m = 2^m + 3^m + 10^m + 11^m \qquad (m = 1, 2, 3).$$

The first example is called a second-order multigrade; the second example, a third-order multigrade.

Adding, subtracting, multiplying and dividing do not affect the equality of a multigrade, provided we perform the same operation or operations on each element in it. For example, Eq. (1) above becomes

$$2^m + 7^m + 9^m = 3^m + 5^m + 10^m,$$

where $m = 1,2,$ if we add 1 to each element; Eq. (2) becomes

$$2^m + 10^m + 16^m + 24^m = 4^m + 6^m + 20^m + 22^m,$$

where $m = 1, 2, 3,$ if we multiply each element by 2.

This note is concerned with what I call second-order Fibonacci multi-multigrades. (I define [1] a multi-multigrade as a multigrade having three or more "components" as compared with the normal two "components" in a multigrade as in (1) and (2) above.)

Here are some examples of Fibonacci multi-multigrades:

(3) $$0^m + (3\cdot 3)^m + (3\cdot 3)^m = (3\cdot 1^2)^m + (3\cdot 1^2)^m + (3\cdot 2^2)^m = \cdots = \cdots$$

(4) $$0^m + (3\cdot 7)^m + (3\cdot 7)^m = (3\cdot 1^2)^m + (3\cdot 2^2)^m + (3\cdot 3^2)^m = (7\cdot 1^2)^m + (7\cdot 1^2)^m + (7\cdot 2^2)^m$$
$$= (1^2)^m + (4^2)^m + (5^2)^m$$

(5) $$0^m + (3\cdot 19)^m + (3\cdot 19)^m = (3\cdot 2^2)^m + (3\cdot 3^2)^m + (3\cdot 5^2)^m = (19\cdot 1^2)^m + (19\cdot 1^2)^m + (19\cdot 2^2)^m$$
$$= (1^2)^m + (7^2)^m + (8^2)^m$$

(6) $$0^m + (3\cdot 49)^m + (3\cdot 49)^m = (3\cdot 3^2)^m + (3\cdot 5^2)^m + (3\cdot 8^2)^m$$
$$= (49\cdot 1^2)^m + (49\cdot 1^2)^m + (49\cdot 2^2)^m = (2^2)^m + (11^2)^m + (13^2)^m$$

$$0^m + [3(F_{2n+4} - F_n\cdot F_{n+1})]^m + [3(F_{2n+4} - F_n\cdot F_{n+1})]^m = [3F_{n+1}^2]^m + [3F_{n+2}^2]^m + [3F_{n+3}^2]^m$$
$$= [(F_{2n+4} - F_n\cdot F_{n+1})F_1^2]^m + [(F_{2n+4} - F_n\cdot F_{n+1})F_2^2]^m + [(F_{2n+4} - F_n\cdot F_{n+1})F_3^2]^m$$
$$= [F_n^2]^m + [(F_{n+5} - F_n)^2]^m + [F_{n+5}^2]^m \qquad (m = 1,2).$$

Clearly, we can expand our multigrades by a simple process. If we multiply (4) by 19×49, (5) by 7×49 and (6) by 7×19, we get

$$0^m + (3\cdot 7\cdot 19\cdot 49)^m + (3\cdot 7\cdot 19\cdot 49)^m = [(3\cdot 19\cdot 49)1^2]^m + [(3\cdot 19\cdot 49)2^2]^m + [(3\cdot 19\cdot 49)3^2]^m$$
$$= [(7\cdot 19\cdot 49)1^2]^m + [(7\cdot 19\cdot 49)1^2]^m + [(7\cdot 19\cdot 49)2^2]^m = [(19\cdot 49)1^2]^m + [(19\cdot 49)4^2]^m + [(19\cdot 49)5^2]^m$$
$$= [(3\cdot 7\cdot 49)2^2]^m + [(3\cdot 7\cdot 49)3^2]^m + [(3\cdot 7\cdot 49)5^2]^m = \cdots = [(7\cdot 49)1^2]^m + [(7\cdot 49)7^2]^m + [(7\cdot 49)8^2]^m$$
$$= [(3\cdot 7\cdot 19)3^2]^m + [(3\cdot 7\cdot 19)5^2]^m + [(3\cdot 7\cdot 19)8^2]^m = \cdots = [(7\cdot 19)2^2]^m + [(7\cdot 19)11^2]^m + [(7\cdot 19)13^2]^m,$$

where $m = 1,2.$

It is possible to obtain multigrades of higher and higher powers by using the traditional method summarized by J.A.H. Hunter and myself in an article several years ago [2].

I give here, by way of example, the following which I recently derived:

$$(F_n^2)^m + [(F_{n+4} - F_n)^2]^m + [3F_{n+2}^2 + 2F_n \cdot F_{n+4} - F_n^2]^m + [F_{n+4}^2 + 3F_{n+2}^2 - F_n^2]^m$$
$$= (3F_{n+1}^2)^m + (2F_n \cdot F_{n+4})^m + (3F_{n+3}^2)^m + (3F_{n+3}^2 + 2F_n \cdot F_{n+4} - 3F_{n+1}^2)^m ,$$

where $m = 1, 2, 3,$

$$0^m + (F_{n+5})^m + (F_{n+5} + F_n)^m + (2F_{n+5} + F_n)^m = (F_{n+2})^m + (F_{n+3})^m + (F_{n+6} + F_n)^m + (F_{n+6} + F_{n+2})^m ,$$

where $m = 1, 2, 3$*.

$$0^m + (F_{n+5} + F_n)^m + (F_{n+5} + F_{n+2})^m + (F_{n+5} + F_{n+3})^m + (F_{n+7} + F_n)^m + (F_{n+7} + F_{n+2})^m$$
$$= (F_{n+2})^m + (F_{n+3})^m + (2F_{n+5})^m + (3F_{n+5} + F_n)^m + (F_{n+6} + F_n)^m + (F_{n+6} + F_{n+2})^m ,$$

where $n = 1, 2, 3, 4$**.

REFERENCES

1. Donald Cross, "Second- and Third-Order Multi-multigrades," *Journal of Recreational Math.,* Vol. 7, No. 1, Winter 1974, pp. 41–44.
2. D.C. Cross, "Multigrades," *Recreational Mathematics Magazine,* No. 13, Feb. 1963, pp. 7–9.
3. D.C. Cross, "The Magic of Squares," *Mathematical Gazette,* Vol. XLV, No. 353, October 1961, pp. 224–227 and Vol. L, No. 372, May 1966, pp. 173–174.

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*If we add F_{n+1} to each term, the multigrade reads

$$(F_{n+1})^m + (F_{n+1} + F_{n+5})^m + (F_{n+2} + F_{n+5})^m + (F_{n+2} + 2F_{n+5})^m = (F_{n+3})^m + (F_{n+1} + F_{n+3})^m$$
$$+ (F_{n+2} + F_{n+6})^m + (F_{n+3} + F_{n+6})m,$$

where $m = 1, 2, 3.$

**If we add F_{n+1} to each term, the multigrade reads

$$(F_{n+1})^m + (F_{n+2} + F_{n+5})^m + (F_{n+3} + F_{n+5})^m + (F_{n+1} + F_{n+3} + F_{n+5})^m + (F_{n+2} + F_{n+7})^m + (F_{n+3} + F_{n+7})^m$$
$$= (F_{n+3})^m + (F_{n+1} + F_{n+3})^m + (F_{n+1} + 2F_{n+5})^m + (F_{n+2} + 3F_{n+5})^m + (F_{n+2} + F_{n+6})^m + (F_{n+3} + F_{n+6})^m;$$

where $m = 1, 2, 3, 4.$

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