GENERALIZED CONVOLUTION ARRAYS

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1. INTRODUCTION

Let

$$\left\{a_n\right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{b_n\right\}_{n=1}^{\infty}$$

be any two sequences, then the Cauchy convolution of the two sequences is a sequence $\left\{c_n\right\}_{n=1}^{\infty}$ whose terms are given by the rule

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1} . \tag{1.1}$$

When we convolve a sequence with itself n times we obtain a new sequence called the n^{th} convolution sequence. The rectangular array whose columns are the convolution sequences is called a convolution array where the n^{th} column of the convolution array is the $(n-1)^{st}$ convolution sequence and the first column is the original sequence.

In Figure 1, we illustrate the first four elements of the convolution array relative to the sequence $\left\{u_n\right\}_{n=1}^{\infty}$

$$\begin{array}{ccccc}
u_1 & u_1^2 & u_1^3 & u_1^4 & \cdots \\
u_2 & 2u_1u_2 & 3u_1^2u_2 & 4u_1^3u_2 & \cdots \\
u_3 & 2u_1u_3+u_2^2 & 3u_1^2u_3+3u_1u_2^2 & 4u_1^3u_3+6u_1^2u_2^2 & \cdots \\
u_4 & 2u_1u_4+2u_2u_3 & 3u_1^2u_4+6u_1u_2u_3+u_2^3 & 4u_1^3u_4+12u_1^2u_2u_3+4u_1u_2^3 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Figure 1

Throughout the remainder of this paper, we let

$$R_{mn}(u_1, u_2, \cdots) \equiv R_{mn} \tag{1.2}$$

be the element in the m^{th} row and n^{th} column of the convolution array.

By mathematical induction, it can be shown that

$$R_{1n} = u_1^n , \tag{1.3}$$

$$R_{2n} = nu_1^{n-1}u_2, \tag{1.4}$$

$$R_{3n} = nu_1^{n-1}u_3 + \binom{n}{2} u_1^{n-2}u_2^2 , \tag{1.5}$$

$$R_{4n} = nu_1^{n-1}u_4 + 2\binom{n}{2} u_1^{n-2}u_2u_3 + \binom{n}{3} u_1^{n-3}u_2^3 , \tag{1.6}$$

$$R_{5n} = nu_1^{n-1}u_5 + \binom{n}{2} u_1^{n-2}(u_3^2 + 2u_2u_4) + 3\binom{n}{3} u_1^{n-3}u_2^2u_3 + \binom{n}{4} u_1^{n-4}u_2^4 , \tag{1.7}$$

$$\begin{aligned} R_{6n} = {} & nu_1^{n-1}u_6 + 2\binom{n}{2} u_1^{n-2}(u_2u_5 + u_3u_4) + 3\binom{n}{3} u_1^{n-3}(u_2^2u_4 + u_2u_3^2) \\ & + 4\binom{n}{4} u_1^{n-4}u_2^3u_3 + \binom{n}{5} u_1^{n-5}u_2^5 , \end{aligned} \tag{1.8}$$

(1.9)
$$R_{7n} = nu_1^{n-1}u_7 + \binom{n}{2} u_1^{n-2}(u_4^2 + 2u_3u_5 + 2u_2u_6) + \binom{n}{3} u_1^{n-3}(u_3^3 + 3u_2^2u_5 + 6u_2u_3u_4)$$
$$+ \binom{n}{4} u_1^{n-4}(4u_2^3u_4 + 6u_2^2u_3^2) + 5\binom{n}{5} u_1^{n-5}u_2^4u_3 + \binom{n}{6} u_1^{n-6}u_2^6 ,$$

and

(1.10)
$$R_{8n} = nu_1^{n-1}u_8 + 2\binom{n}{2} u_1^{n-2}(u_2u_7 + u_3u_6 + u_4u_5) + 3\binom{n}{3} u_1^{n-3}(u_2^2u_6 + 2u_2u_3u_5 + u_2u_4^2 + u_3^2u_4)$$
$$+ 4\binom{n}{4} u_1^{n-4}(u_2^3u_5 + 3u_2^2u_3u_4 + u_2u_3^3) + 5\binom{n}{5} u_1^{n-5}(u_2^4u_4 + 2u_2^3u_3^2)$$
$$+ 6\binom{n}{6} u_1^{n-6}u_2^5u_3 + \binom{n}{7} u_1^{n-7}u_2^7 .$$

The purpose of this article is to examine the general expression for R_{mn} and to find a formula for the generating function for any row of the convolution array.

2. PARTITIONS OF m AND R_{mn}

A partition of a nonnegative integer m is a representation of m as a sum of positive integers called parts of the partition. The function $\pi(m)$ denotes the number of partitions of m.

The partitions of the integers one through seven are given in Table 1.

Table 1

m	Partitions of m	$\pi(m)$
1	1	1
2	2, 1 + 1	2
3	3, 1 + 2, 1 + 1 + 1	3
4	4, 2 + 2, 1 + 3, 1 + 1 + 2, 1 + 1 + 1 + 1	4
5	5, 2 + 3, 1 + 4, 1 + 1 + 3, 1 + 2 + 2, 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 1	7
6	6, 3 + 3, 2 + 4, 1 + 5, 2 + 2 + 2, 1 + 1 + 4, 1 + 2 + 3, 1 + 1 + 1 + 3, 1 + 1 + 2 + 2, 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 1 + 1	11
7	7, 1 + 6, 2 + 5, 3 + 4, 1 + 1 + 5, 1 + 2 + 4, 1 + 3 + 3, 2 + 2 + 3, 1 + 1 + 1 + 4, 1 + 1 + 2 + 3, 1 + 2 + 2 + 2, 1 + 1 + 1 + 1 + 3, 1 + 1 + 1 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 1 + 1 + 1	15

Comparing the partitions of m, for $m = 1$ through $m = 7$, with the expressions for R_{mn} it appears as if the following are true.

1. The number of terms in R_{mn} is equal to $\pi(m-1)$.
2. The number of expressions whose coefficient is $\binom{n}{j}$, for $j = 1, 2, \cdots, m-1$, is the number of partitions of $m-1$ into j parts.
3. The power of u_{t+1} in an expression is the same as the number of times t occurs in the partition of $m-1$.
4. The numerical coefficient of an expression involving $\binom{n}{j}$, for $j = 1, 2, 3, \cdots, m-1$, is equal to the product of the factorials of the exponents of the terms of the sequence
$$\left\{ u_n \right\}_{n=1}^{\infty}$$
in the expression divided into j factorial. The exponent for u_1 is not included in the product.

In [4], it is shown that these are in fact true statements. That is,

(2.1)
$$R_{mn}(u_1, u_2, \cdots) = \sum_{k=1}^{m-1} \binom{n}{k} u_1^{n-k} P_{mk}(u_1, u_2, \cdots),$$

where

(2.2)
$$P_{mk}(u_1, u_2, u_3, \cdots) = \sum_{\pi(m-1)} \frac{k!}{a_2!a_3!\cdots a_{m-1}!} u_2^{a_2}u_3^{a_3}\cdots u_m^{a_m}, \quad k = a_2 + a_3 + \cdots + a_m .$$

3. SOME FINITE DIFFERENCES

The first difference of a function $f(x)$ is defined as

$$\Delta f(x) = f(x+1) - f(x). \tag{3.1}$$

In an analogous fashion, we define recursively the n^{th} difference $\Delta^n f(x)$ of $f(x)$ as

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)). \tag{3.2}$$

In [3], we find

$$\sum_{x=0}^{m-1} (-1)^x \binom{m-1}{x} f(x) = (-1)^{m-1}\Delta^{m-1} f(0). \tag{3.3}$$

Using mathematical induction, it is easy to show the following.

Theorem 3.1. If $f(x) = \binom{r-x+s}{j}$ then $\Delta^n f(x) = (-1)^n \binom{r-x+s-n}{j-n}$

and

Theorem 3.2. If $f(x) = \binom{r+x+s}{j}$ then $\Delta^n f(x) = \binom{r+x+s}{j-n}$.

Applying (3.3), we then have

Theorem 3.3. If $f(x) = \binom{r-x+s}{j}$ then

$$\sum_{x=0}^{m-1} (-1)^x \binom{m-1}{x}\binom{r-x+s}{j} = \binom{r+s-m+1}{j-m+1}.$$

and

Theorem 3.4. If $f(x) = \binom{r+x+s}{j}$ then

$$\sum_{x=0}^{m-1} (-1)^x \binom{m-1}{x}\binom{r+x+s}{j} = (-1)^{m-1}\binom{r+s}{j-m+1}.$$

4. THE MAIN THEOREM

Combining (2.1) with Theorem 3.3., we see that, whenever $u_1 = 1$, we then have

$$\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} R_{m,n-k+1} = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \sum_{j=1}^{m-1} \binom{n-k+1}{j} P_{mj}$$

$$= \sum_{j=1}^{m-1} P_{mj} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k}\binom{n-k+1}{j} = \sum_{j=1}^{m-1} P_{mj}\binom{n-m+2}{j-m+2} = P_{m,m-1}.$$

Now, the only way to partition $m-1$ into $m-1$ parts is to let every part of the partition equal one. Hence, by (2.2), we have

$$P_{m,m-1} = u_2^{m-1}$$

so that

$$\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} R_{m,n-k+1} = u_2^{m-1}. \tag{4.1}$$

From (4.1), it is easy to see that the generating function $g_m(x)$ for the sequence $\{R_{m,n+1}\}_{n=0}^{\infty}$, where $u_1 = 1$, is of the form

$$g_m(x) = \frac{h_m(x)}{(1-x)^m} = \sum_{n=0}^{\infty} R_{m,n+1}x^n. \tag{4.2}$$

In order to determine the generating function $g_m(x)$ for the m^{th} row of the convolution array, it is necessary to determine what is commonly called "Pascal's attic." That is, we need to know the values for the columns corresponding to the negative integers and zero subject to the condition of (4.1). With this in mind, we develop the next two theorems.

Theorem 4.1. If $m \geq 2$ and $u_1 = 1$ then $R_{m,0} = 0$.

Proof. Letting $n = m-2$ in (4.1), we have

$$(-1)^{m-1}R_{m,o} = \sum_{k=0}^{m-2} (-1)^{k+1}\binom{m-1}{k}R_{m,m-k+1} + u_2^{m-1} = \sum_{k=1}^{m-1} (-1)^{m-k}\binom{m-1}{m-k-1} R_{mk}$$

$$+ u_2^{m-1} = \sum_{k=1}^{m-1} (-1)^{m+k}\binom{m-1}{k}R_{mk} + u_2^{m-1}\ .$$

By (2.1), using j as the variable of summation, and Theorem 3.4 with $r = s = 0$, we obtain

$$(-1)^{m-1}R_{m,0} = \sum_{k=1}^{m-1} (-1)^{m+k}\binom{m-1}{k}\sum_{j=1}^{m-1}\binom{k}{j}P_{mj} + u_2^{m-1}$$

$$= (-1)^m \sum_{j=1}^{m-1} P_{mj}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\binom{k}{j} + u_2^{m-1}$$

$$= -\sum_{j=1}^{m-1} P_{mj}\binom{o}{j-m+1} + u_2^{m-1} = -P_{m,m-1} + u_2^{m-1} = 0$$

and the theorem is proved.

Theorem 4.2. If $n \geq 1$, $m \geq 2$ and $u_1 = 1$ then

$$R_{m,-n} = \sum_{k=1}^{m-1} (-1)^k \binom{n+k-1}{k} P_{mk}\ .$$

Proof. We shall use the strong form of mathematical induction.

Replacing n by $m-3$ in (4.1) and following the argument of Theorem 4.1 where we let $r = 0$ and $s = -1$ in Theorem 3.4, we have

$$(-1)^{m-1}R_{m,-1} = \sum_{k=0}^{m-2}(-1)^{k+1}\binom{m-1}{k}R_{m,m-k-2} + u_2^{m-1} = \sum_{k=1}^{m-1}(-1)^{m+k}\binom{m-1}{k}R_{m,k-1} + u_2^{m-1}$$

$$= (-1)^m \sum_{j=1}^{m-1} P_{mj}\sum_{k=1}^{m-1}(-1)^k\binom{m-1}{k}\binom{k-1}{j} + u_2^{m-1}$$

$$= (-1)^m \sum_{j=1}^{m-1} P_{mj}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\binom{k-1}{j} - (-1)^m\sum_{j=1}^{m-1}P_{mj}\binom{-1}{j} + u_2^{m-1}$$

$$= -\sum_{j=1}^{m-1} P_{mj}\binom{-1}{j-m+1} - (-1)^m\sum_{j=1}^{m-1}P_{mj}\binom{-1}{j} + u_2^{m-1}\ .$$

Recalling that

$$\binom{-n}{m} = (-1)^m\binom{n+m-1}{m}$$

if $n \geq 1$, and $m \geq 0$ and $\binom{n}{-m} = 0$ for all n provided $m \geq 1$, we have

$$(-1)^{m-1}R_{m,-1} = -P_{m,m-1} - (-1)^m \sum_{j=1}^{m-1}(-1)^j P_{mj} + u_2^{m-1}$$

so that

$$R_{m,-1} = \sum_{j=1}^{m-1} (-1)^j P_{mj}$$

and the theorem is true for $n = 1$.

We now assume that the theorem is true for all positive integers less than or equal to t. Replacing n by $m-t-3$ in (4.1), we see that

$$\begin{aligned}(-1)^{m-1}R_{m,-(t+1)} &= \sum_{k=0}^{m-2} (-1)^{k+1}\binom{m-1}{k} R_{m,m-t-k-2} + u_2^{m-1}\\ &= \sum_{k=1}^{m-1} (-1)^{m+k}\binom{m-1}{k} R_{m,-(t-k+1)} + u_2^{m-1}\\ &= \sum_{j=1}^{m-1} (-1)^{m+j} P_{mj} \sum_{k=1}^{m-1} (-1)^k \binom{m-1}{k}\binom{t-k+j}{j} + u_2^{m-1}\end{aligned},$$

where the last equation is obtained by the induction hypothesis.

Multiplying by $(-1)^{m-1}$ and introducing $k = 0$, one has

$$\begin{aligned}R_{m,-(t+1)} &= \sum_{j=1}^{m-1} (-1)^{j-1} P_{mj} \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k}\binom{t-k+j}{j} + \sum_{j=1}^{m-1} (-1)^j \binom{t+j}{j} P_{mj} + (-u_2)^{m-1}\\ &= \sum_{j=1}^{m-1} (-1)^{j-1} P_{mj} \binom{t+j-m+1}{j-m+1} + \sum_{j=1}^{m-1} (-1)^j \binom{t+j}{j} P_{mj} + (-u_2)^{m-1}\\ &= \sum_{j=1}^{m-1} (-1)^j \binom{t+j}{j} P_{mj}\end{aligned},$$

where the second equation is obtained by use of Theorem 3.3 with $r = t$ and $s = j$ and the theorem is proved.

We are now in a position to calculate the generating function for the m^{th} row of a convolution array when $u_1 = 1$.

When $m = 1$, we see that $R_{1,n} = 1$ for all $n \geqslant 0$ so that

$$g_1(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{4.3}$$

By (4.1), we have

$$R_{m,n+1} = \sum_{k=1}^{m-1} (-1)^{k+1}\binom{m-1}{k} R_{m,n-k+1} + u_2^{m-1}$$

so that when $m \geqslant 2$, we can use (4.2) to obtain

$$g_m(x) = \sum_{n=0}^{\infty}\sum_{k=1}^{m-1} (-1)^{k+1}\binom{m-1}{k} R_{m,n-k+1}x^n + \sum_{n=0}^{\infty} u_2^{m-1}x^n = \sum_{k=1}^{m-1} (-1)^{k+1}\binom{m-1}{k} x^k \times$$

$$\left(\sum_{n=0}^{\infty} R_{m,n-k+1}x^{n-k} + \frac{u_2^{m-1}}{1-x}\right) = \sum_{k=1}^{m-1} (-1)^{k+1}\binom{m-1}{k} x^k\left(g_m(x) + \sum_{n=1}^{k-1} R_{m,-n}x^{-n-1}\right) + \frac{u_2^{m-1}}{1-x}.$$

Hence,

$$g_m(x) = \frac{(1-x)\displaystyle\sum_{k=1}^{m-1}\sum_{n=1}^{k-1}(-1)^{k+1}\binom{m-1}{k}R_{m,-n}x^{k-n-1} + u_2^{m-1}}{(1-x)^m}, \quad m \geqslant 2\,. \tag{4.4}$$

For special sequences

$$\{u_n\}_{n=1}^{\infty}$$

with $u_1 = 1$, the polynomial in the numerator of $g_m(x)$, $m \geqslant 1$, is predictable from the convolution array of the sequence. This matter will be covered by the authors in another paper which will appear in the very near future.

REFERENCES

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LETTER TO THE EDITOR

February 20, 1975

Dear Mr. Hoggatt:

I'm afraid there was an error in the February issue of *The Fibonacci Quarterly.* Mr. Shallit's proof that phi is irrational is correct up to the point where he claims that $1/\phi$ can't be an integer. He has no basis for making that claim, as ϕ was defined as a rational number, not an integer.

The proof can, however, be salvaged after the point where p is shown to equal 1. Going back to the equation $p^2 - pq = q^2$, we can add pq to each side, and factor out a q from the right: $p^2 = q(q+p)$. Using analysis similar to Mr. Shallit's, we find that q must also equal 1. Therefore, $\phi = p/q = 1/1 = 1$. However, $\phi^2 - \phi - 1 = -1 \neq 0$; thus, our assumption was false, and ϕ is irrational.

Sincerely,
s/David Ross, Student,
Swarthmore College

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