# ON LUCAS NUMBERS WHICH ARE ONE MORE THAN A SQUARE

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Let $F_n$ be the $n^{th}$ term in the Fibonacci sequence, defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n,$$

and let $L_m$ be the $n^{th}$ term in the Lucas sequence, defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

In a previous paper [4], the author proved that the only numbers in the Fibonacci sequence of the form $y^2 + 1$ are

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2 \quad \text{and} \quad F_5 = 5.$$

The purpose of the present paper is to prove the corresponding result for Lucas numbers. In particular, we prove the following:

***Theorem.*** The only numbers in the Lucas sequence of the form

$$y^2 + 1, \quad y \in z, \quad y \geqslant 0$$

are $L_0 = 2$ and $L_1 = 1$.

In the course of our investigations, we shall require the following results, some of which were proved by Cohn [1], [2], [3].

(1) $$L_{2n} = L_n^2 + 2(-1)^{n-1}.$$

(2) $$(F_{3n}, L_{3n}) = 2 \quad \text{and} \quad (F_n, L_n) = 1 \text{ if } 3 \nmid n.$$

(3) $$L_n^2 - 5F_n^2 = 4(-1)^n.$$

(4) If $F_{2n} = x^2$, $n > 0$, then $2n = 0, 2$ or $12$.

(5) The only non-negative solutions of the equation $x^2 - 5y^4 = 4$ are

$$[x,y] = [2,0], [3,1] \text{ and } [322,12].$$

(6) $L_n$ is never divisible by 5 for any $n$.

(7) $$\text{If} \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} \quad \text{then} \quad F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

(8) $$F_{2n} = F_n L_n.$$

(9) If $L_n = x^2$, $n > 0$, then $n = 1$ or $3$.

(10) If $L_n = 2x^2$, $n > 0$, then $n = 0$ or $6$.

We now return to the proof of our theorem, and consider two cases,

CASE I. $n$ even: If $L_{2n} = y^2 + 1$, then by (1), either

$$y^2 + 1 = L_n^2 + 2 \quad \text{or} \quad y^2 + 1 = L_n^2 - 2.$$

The first case yields

$$L_n^2 - y^2 = -1, \quad L_n = 0, \quad y = 1,$$

which is impossible. The second case yields

$$L_n^2 - y^2 = 3,$$

and it is easily proved that the only integer solution of this equation is

$$L_n = 2, \quad y = 1.$$

CASE II. $n$ odd: First, we prove the following Lemmas:

*Lemma 1.* If $F_{2n} = 5x^2$ then $n = 0$.

*Proof.* By (8), we have $F_n L_n = 5x^2$ and, by (2), either

$$(F_n, L_n) = 1 \quad \text{or} \quad (F_n, L_n) = 2.$$

If $(F_n, L_n) = 1$, then, by (6),

$$F_n = 5s^2, \qquad L_n = t^2.$$

But then $n = 1$ or $3$ and $F \neq 5s^2$. If $(F_n, L_n) = 2$, then we conclude that

$$F_n = 10s^2, \qquad L_n = 2t^2.$$

By (10), $n = 0$ or $6$. But $F_n = 10s^2$ only for $n = 0$.

*Lemma 2.* The only integer solution of the equation $u^2 - 125v^4 = 4$ is

$$u = \pm 2, \quad v = 0.$$

*Proof.* If $u^2 - 125v^4 = 4$, then $u$ and $5v^2$ are a set of solutions of

$$p^2 - 5q^2 = 4$$

thus

$$u + 5v^2\sqrt{5} = 2\left(\frac{3+\sqrt{5}}{2}\right)^n = 2\alpha^{2n}, \qquad u - 5v^2\sqrt{5} = 2\beta^{2n}.$$

so $F_{2n} = 5v^2$ and thus $v = 0$.

Now let us use (3) with $n$ odd and $L_n = y^2 + 1$. We get

(11) $$(y^2 + 1)^2 + 4 = 5x^2,$$

and we wish to show that the only integer solution of this equation is $y = 0$, $x = 1$. Note first that if $y$ is odd the equation is impossible mod 16.

On factorizing (11) over the Gaussian integers, we set

$$(y^2 + 1 + 2i)(y^2 + 1 - 2i) = 5x^2.$$

Since $y$ is even, the two factors on the left-hand side of this equation are relatively prime. Thus we conclude

$$y^2 + 1 + 2i = (1 + 2i)(a + bi)^2.$$

This yields

$$a^2 + ab - b^2 = 1, \qquad a^2 - 4ab - b^2 = y^2 + 1,$$

i.e.,

(12) $$a^2 + ab - b^2 = 1$$

and

$$5ab = -y^2.$$

The first equation of (12) yields $(a,b) = 1$, and it may be written

(13) $$(2a + b)^2 - 5b^2 = 4.$$

Since $(a,b) = 1$ the second equation of (12) yields either

(14) $$b = \pm t^2, \qquad a = \mp 5a^2$$

or

(15) $$b = \pm 5t^2, \qquad a = \mp s^2 \;.$$

Equations (13) and (14) yield

$$(\mp 10s^2 \pm t^2)^2 - 5t^4 = 4.$$

By (5), the only integer solutions of this equation occur for $t = 0$, 1 or 12. But none of these values of $t$ yield a value for $s$. Equations (13) and (15) yield

$$(\mp 2s^2 \pm 5t^2)^2 - 125t^4 = 4.$$

By Lemma 2, $t = 0$, $s = 1$, $a = \pm 1$, $b = 0$, $L_n = 1$. The proof is complete.

## REFERENCES

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2. J. H. E. Cohn, "Square Fibonacci Numbers, Etc.," *The Fibonacci Quarterly*, Vol. 2, No. 2 (April 1964), pp. 109–113.
3. J. H. E. Cohn, "Lucas and Fibonacci Numbers and Some Diophantine Equations," *Proc. Glasgow Math. Assoc.*, 7 (1965), pp. 24–28.
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[Continued from P. 339.]

Since

$$(a/-1) = (b/-1) = 1,$$

therefore

$$\begin{aligned}(-a/-b)(-b/-a) &= (a/b)(b/a)(-1/a)(-1/b)\\ &= ((-1/a)/(-1/b))(-1/a)(-1/b)\\ &= 1\end{aligned}$$

if and only if

$$(-1/a) = (-1/b) = 1\,.$$

Therefore,

(4) $$(-a/-b)(-b/-a) = -((-1/-a)/(-1/-b)).$$

From (1), (2), (3) and (4), it can be seen that the theorem is true for all sixteen combinations of

$$(a/-1) = \pm 1, \qquad (b/-1) = \pm 1, \qquad (-1/a) = \pm 1 \quad \text{and} \quad (-1/b) = \pm 1.$$

*Corollary 1.* If $a \equiv 0$ or 1 (mod 2), $b \equiv 1$ (mod 2) and $(a,b) = 1$, and if $a_1 \equiv a_2$ (mod $b$), then

$$(a_1 a_2/b) = \left(\frac{(a_1 a_2/-1)}{(b/-1)}\right) \;.$$

In other words, $(a_1 a_2/b) = 1$ if and only if $a_1 a_2$ is positive and/or $b$ is positive.

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