ON THE GENERALIZATION OF THE FIBONACCI NUMBERS

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I.

The Fibonacci numbers ($F_0 = F_1 = 1$; $F_n = F_{n-1} + F_{n-2}$, if $n \geq 2$) are very useful in describing the ladder-network of Fig. 1, if $r = R$ (cf. [1], [2], [3]). If the common value of the resistances R and r is chosen to be unity, the resistance Z_n of the ladder-network can be calculated on the following way:

(1a) $$Z_n = \frac{F_{2n}}{F_{2n-1}} .$$

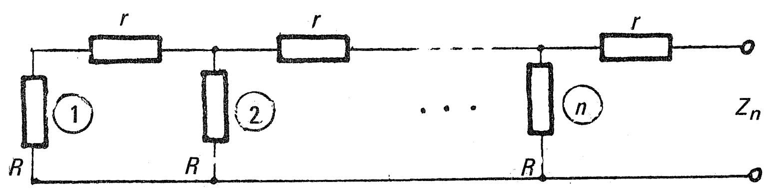

Figure 1

Let $R \neq r$. For the sake of convenient notation let $x = r/R$ and $z_n = Z_n/R$. Then

(1) $$z_n = \frac{f_{2n}(x)}{f_{2n-1}(x)} ,$$

where $f_0(x) = f_1(x) = 1$; and for $n \geq 2$,

(*) $$f_n(x) = \begin{cases} f_{n-1}(x) + f_{n-2}(x) & \text{if } n \text{ is odd} \\ xf_{n-1}(x) + f_{n-2}(x) & \text{if } n \text{ is even} \end{cases} .$$

This fact gave us the idea to examine into the sequences, defined by a finite number of homogeneous linear recurrences which are to be used cyclically. We may assume without loss of generality that the length of the recurrences are equal and that this common length $\underline{m}$ equals the number of the recurrences:

$$f_n = \begin{cases} a_1^0 f_{n-1} + \cdots + a_m^0 f_{n-m} & \text{if } n \equiv 0 \pmod m \\ a_1^1 f_{n-1} + \cdots + a_m^1 f_{n-m} & \text{if } n \equiv 1 \pmod m \\ \vdots & \\ a_1^{m-1} f_{n-1} + \cdots + a_m^{m-1} f_{n-m} & \text{if } n \equiv m-1 \pmod m \end{cases}$$

It has been proved in [5] that the same sequence f_n can be generated by a certain unique recurrence too, which has length m^2 and "interspaces" of length $\underline{m}$, i.e.,

$$f_n = b_1 f_{n-m} + b_2 f_{n-2m} + \cdots + b_m f_{n-m^2} .$$

Applying our results to (*) we have

$$f_n(x) = (2+x)\cdot f_{n-2}(x) - f_{n-4}(x),$$

or, after the calculation of the generating function and expanding it into Taylor-series,

$$(2)\qquad f_n(x) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} x^{[n/2]-i}.$$

This enables us to solve not only the problem of the lumped network mentioned above, but a special question of the theory of the distributed networks (e.g., transmission lines) can also be solved. If we want to describe the pair of transmission lines having resistance r_0 and shunt-admittance $1/R_0$ (see Fig. 2), then put $r = r_0/n$ and $R = R_0 \cdot n$. Applying (1) and (2) we have

$$Z_n^* = \frac{r_0}{n}\,\frac{g_n\left(\dfrac{R_0 n^2}{r_0}\right)}{h_n\left(\dfrac{R_0 n^2}{r_0}\right)},$$

where

$$g_n(x) = \sum_{j=1}^{n} \binom{2n-j}{j-1}\cdot x^j \qquad \text{and} \qquad h_n(x) = \sum_{j=0}^{n} \binom{2n-j}{j}\cdot x^j.$$

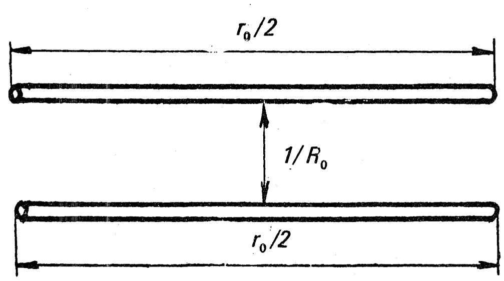

Figure 2

The following simultaneous system of recurrences can be found:

$$g_{n-1}(x) = h_n(x) - (1+x)h_{n-1}(x)$$

$$x^2\cdot h_{n-2}(x) = g_n(x) - (1+x)g_{n-1}(x),$$

which enables us to give an explicit form to $g_n(x)$ and $h_n(x)$. At last

$$\lim Z_n^* = \sqrt{R_0 r_0}\cdot th\sqrt{r_0/R_0}, \quad \text{if} \quad n \to \infty,$$

where

$$th\, y = \frac{e^y - e^{-y}}{e^y + e^{-y}}.$$

This is exactly the result, which can also be received from a system of partial differential equations (the telegraph-equations).

II.

On the other hand, (2) can be considered as a generalization of the Fibonacci sequence. Trivially $f_n(1) = F_n$; and

$$F_n = \sum_{i=0}^{[n/2]} \binom{n-i}{i} ,$$

a well-known result about the Fibonacci numbers. Similarly,

$$F_n = 3F_{n-2} - F_{n-4} ,$$

if $n \geqslant 4$, or

$$F_n = tF_{n-3} + (17-4t)F_{n-6} + (4-t)F_{n-9}$$

for any t, if $n \geqslant 9$ and an infinite number of longer recurrences (length m^2 and interspaces m for arbitrary $m = 2,3,\cdots$) could be similarly produced.

A possible further generalization of the Fibonacci numbers is

$$F_{n,p}(x) = \sum_{i=0}^{\left[\frac{n}{p+1}\right]} \binom{n-ip}{i} x^{\left[\frac{n}{p+1}\right]-i} ,$$

where p is an arbitrary non-negative integer.

This definition is the generalization of the $u(n;p,1)$ numbers of [4]. The following recurrence can be proved for the $F_{n,p}(x)$ polynomials:

$$xF_{n-p-1,p}(x) = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} F_{n-(p+1)i,p}(x) . \tag{3}$$

Similarly, it can be easily proved that the generating function

$$\sum_{i=0}^{\infty} F_{i,p}(x)\cdot z^i$$

has the following denominator:

$$(1-z^{p+1})^{p+1} - x\cdot z^{p+1} .$$

As a last remark, it is to be mentioned that a further generalization of the functions $F_{n,p}(x)$ can be given (cf. [4]):

$$F_{n;p,q}(x) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n-ip}{iq} x^{\left[\frac{n}{p+q}\right]-i} ;$$

but this case is more difficult. A recurrence, similar to (3) can be found, which contains on the left side the higher powers of x, too. However, essentially new problems arise considering the case $q \geqslant 2$.

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