MINIMUM SOLUTIONS TO $x^2 - Dy^2 = \pm 1$

GREGORY WULCZYN
Bucknell University, Lewisburg, Pennsylvania 17837

A solution pair (x_0, y_0) to $x^2 - Dy^2 = \pm 1$ shall be considered a minimum solution if y_0 is minimum. Throughout this article F_n stands for the n^{th} Fibonacci number of

$$\{1, 1, 2, 3, \cdots, F_{n+2} = F_{n+1} + F_n, \qquad F_1 = F_2 = 1\} \;;$$

$$F_n = \frac{1}{\sqrt{5}}(a^n - b^n), \qquad a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2}, \qquad ab = -1.$$

I. CONTINUED FRACTION EXPANSION OF ODD PERIOD $2r+1$, $r \not\equiv 0 \pmod 3$, $r > 1$

Let $D = m^2 + k$, $m \leqslant k \leqslant 2m$, and assume a continued fraction expansion all of whose middle elements are ones, thus assuring minimum y.

$$y = F_{2r}, \qquad x = mF_{2r} + F_{2r-1}$$

$$(mF_{2r} + F_{2r-1})^2 - y^2(m^2 + k) = 1$$

upon using

$$F_{t+1}F_{t-1} - F_t^2 = (-1)^t$$

simplifies to

(1)
$$2mF_{2r-1} - kF_{2r} = -F_{2r-2}\,.$$

This has integer solutions given by

$$m = sF_{2r} + \tfrac{1}{2}(F_{2r} + 1),$$

$$k = 2sF_{2r-1} + F_{2r-1} + 1, \qquad D = s^2F_{2r}^2 + s(F_{2r}^2 + 2F_{2r} + F_{2r-3}) + \tfrac{1}{4}(F_{2r} + 1)^2 + F_{2r-1} + 1\,.$$

$x^2 - Dy^2 = 1$ has integer solutions given by

$$x = sF_{2r}^2 + \tfrac{1}{2}F_{2r}(F_{2r} + 1) + F_{2r-1}, \qquad y = F_{2r}\,.$$

II. CONTINUED FRACTION EXPANSION OF ODD PERIOD $6r+1$

Let $D = m^2 + k$. Assume that the central integer in the continued fraction expansion is 2 and that all the other middle elements are ones. The half period expansion is:

		$(3r-1)$ ones			
m	1	1	1	1	2
m	$m+1$	$2m+1$	$mF_{3r-1} + F_{3r-2}$	$mF_{3r} + F_{3r-1}$	———
1	1	2	F_{3r-1}	F_{3r}	$2F_{3r} + F_{3r-1}$

$$y = F_{3r}[F_{3r-1} + 2F_{3r} + F_{3r-1}] = 2F_{3r}F_{3r+1}$$

$$x = 2mF_{3r}F_{3r+1} + F_{3r}^2 + F_{3r-1}F_{3r+1}$$

$$[2mF_{3r}F_{3r+1} + F_{3r}^2 + F_{3r-1}F_{3r+1}]^2 - 4F_{3r}^2F_{3r+1}^2(m^2 + k) = 1$$

simplifies to

$$m(F_{3r}^2 + F_{3r-1}F_{3r+1}) - F_{3r}F_{3r+1}k = -F_{3r}F_{3r-1}$$

(2)
$$k = m + \frac{F_{3r-1}(mF_{3r-1} + F_{3r})}{F_{3r}F_{3r+1}}$$

(a) $$r = 2u + 1.$$

Equation (2) transformed has integer solutions for m and k given by

$$m = F_{6u+3}F_{6u+4}s + F_{6u+3}^2$$

$$k = (2F_{6u+3}^2 - 1)s + F_{6u+1}F_{6u+2} + F_{6u+3}^2.$$

(b) $$r = 2u$$

Equation (2) transformed has integer solutions for m and k given by

$$m = F_{6u}F_{6u+1}s + F_{6u}F_{6u-1},$$

$$k = (2F_{6u}^2 + 1)s + F_{6u-1}F_{6u-3} + F_{6u}F_{6u-1}$$

III. CONTINUED FRACTION EXPANSIONS OF EVEN PERIOD $2r + 2,\ r \not\equiv 1 \pmod 3,\ r \geq 1,\ D = m^2 + k$

Assume a continued fraction expansion all of whose middle elements are ones, thus assuring minimum y.

$$y = F_{2r+1}, \qquad x = mF_{2r+1} + F_{2r} \qquad \text{and} \qquad (mF_{2r+1} + F_{2r})^2 - F_{2r+1}^2(m^2 + k) = -1$$

simplifies to

(3) $$2mF_{2r} - kF_{2r+1} = -F_{2r-1}$$

Equation (3) has integer solutions given by

$$m = sF_{2r+1} + \tfrac{1}{2}(F_{2r+1} + 1), \qquad k = 2sF_{2r} + F_{2r} + 1,$$

$$D = s^2F_{2r+1}^2 + s(F_{2r+1}^2 + F_{2r+1} + 2F_{2r}) + \tfrac{1}{4}(F_{2r+1} + 1)^2 + F_{2r} + 1.$$

$x^2 - Dy^2 = -1$ has integer solutions given by

$$x = sF_{2r+1}^2 + \tfrac{1}{2}F_{2r+1}(F_{2r+1} + 1) + F_{2r}, \qquad y = F_{2r+1}.$$

IV. CONTINUED FRACTION EXPANSIONS WITH EVEN PERIOD $6r - 2,\ r \geq 1$

Let $D = m^2 + k$ and assume that the two central elements are each two and the other middle elements are all ones. From the half period expansion:

		$(3r-3)$ ones		
m	1	1 ---------- 1		2
$\frac{m}{1}$	$\frac{m+1}{1}$	$\frac{2m+1}{2}$	$\frac{mF_{2r-2} + F_{3r-3}}{F_{3r-2}}$	$\frac{mF_{3r} + F_{3r-1}}{F_{3r}}$

$$y = F_{3r-2}^2 + F_{3r}^2$$

$$x = (F_{3r-2}^2 + F_{3r}^2)m + F_{3r-2}F_{3r-3} + F_{3r}F_{3r-1}$$

$$(my + F_{3r-2}F_{3r-3} + F_{3r}F_{3r-1})^2 - y^2(m^2 + k) = -1$$

simplifies to

$$2(F_{3r-2}F_{3r-3} + F_{3r}F_{3r-1})m - (F_{3r}^2 + F_{3r-2}^2)k = -F_{3r-3}^2 - F_{3r-1}^2$$

(4) $$k = m + \frac{m(F_{3r}F_{3r-3} + F_{3r-2}F_{3r-5}) + F_{3r-3}^2 + F_{3r-1}^2}{F_{3r}^2 + F_{3r-2}^2}$$

(a) $$r = 2u$$

$$k = m + \frac{m(F_{6u}F_{6u-3} + F_{6u-2}F_{6u-5}) + F_{6u-3}^2 + F_{6u-1}^2}{F_{6u}^2 + F_{6u-2}^2}$$

has integer solutions given by

$$m = \tfrac{1}{2}(F_{6u}F_{6u-3} + F_{6u-2}F_{6u-5} + 1),$$

$$k = m + \tfrac{1}{2}(F_{6u-3}^2 + F_{6u-5}^2 + 1)$$

(b) $$r = 2u + 1$$

$$k = m + \frac{m(F_{6u}F_{6u+3} + F_{6u+1}F_{6u-2}) + F_{6u}^2 + F_{6u+2}^2}{F_{6u+3}^2 + F_{6u+1}^2}$$

has integer solutions given by

$$m = -\tfrac{1}{2}(F_{6u}F_{6u+3} + F_{6u+1}F_{6u-2} - 1), \qquad k = m - \tfrac{1}{2}(F_{6u}^2 + F_{6u-2}^2 - 1).$$

MINIMUM SOLUTION TABLE

period	D	x	y
2	$m^2 + 1 : 2$	$m : 1$	1
3	$m^2 + 2m : 3$	$m + 1 : 2$	1
4	$25s^2 + 64s + 41 : 41$	$25s + 32 : 32$	5
5	$9s^2 + 16s + 7 : 7$	$9s + 8 : 8$	3
6	13	18	5
7	21	55	12
8	58	99	13
9	135	244	21
10	113	776	73
11	819	1574	55
12	2081	4060	89
13	1650	8449	208
14	13834	27405	233
15	35955	71486	377
16	1370345	1551068	1325
17	244647	488188	987
18	639389	1276990	1597
19	1337765	4325751	3740
20	4374866	8745055	4181
21	11448871	22890176	6765
22	7877105	66688052	23761
23	78439683	156859562	17711
24	205337953	410643864	28657

A continued fraction expansion is unique; has the unique half-period relations

$$q_{2r} = q_r(q_{r-1} + q_{r+1}), \qquad p_{2r} = p_{r-1}q_r + p_rq_{r+1}$$

for period $2r + 1$,

$$q_{2r+1} = q_r^2 + q_{r+1}^2, \qquad p_{2r+1} = p_rq_r + p_{r+1}q_{r+1}$$

for period $2r + 2$; and furnish a minimum primitive solution to $x^2 - Dy^2 = \pm 1$. Using Fibonacci identities it can be shown that all the assumed continued fraction expansions obey the proper half-period relations, give a minimum primitive solution to $x^2 - Dy^2 = \pm 1$ and hence are the actual continued fraction expansions. The half-period relations are explicitly stated as the x and y values in II and IV. The other Fibonacci identities needed are

(1) $$F_{2r}^2 + 1 = F_{2r-1}F_{2r+1}$$

(2) $$F_r^2 + F_{r+1}^2 = F_{2r+1};$$

(3) $$F_{r-1}(F_{r-2} + F_r) = F_{2r-2};$$

(4) $$F_{2n-1}^2 - 1 = F_{2n}F_{2n-2}.$$

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