ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania 17745

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-258 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Sum the series

$$S \equiv \sum x^a y^b z^c t^d,$$

where the summation is over all non-negative a, b, c, d such that

$$\begin{cases} 2a \leqslant b+c+d \\ 2b \leqslant a+c+d \\ 2c \leqslant a+b+d \\ 2d \leqslant a+b+c. \end{cases}$$

H-259 Proposed by R. Finkelstein, Tempe, Arizona.

Let p be an odd prime and m an odd integer such that $m \not\equiv 0 \pmod p$. Let $F_{mp} = F_p \cdot Q$. Can $(F_p, Q) > 1$?

H-260 Proposed by H. Edgar, San Jose State University, San Jose, California.

Are there infinitely many subscripts, n, for which F_n or L_n are prime?
Editorial Note: Good luck on this one!

SOLUTIONS

CORRECTION

H-179 Proposed by D. Singmaster, Bedford College, University of London, England.

Let k numbers $p_1, p_2, \cdots, p_k$ be given. Set $a_n = 0$ for $n < 0$; $a_0 = 1$ and define a_n by the recursion

$$a_n = \sum_{i=1}^{k} p_i a_{n-i} \quad \text{for} \quad n > 0.$$

1. Find simple necessary and sufficient conditions on the p_i for $\lim_{n\to\infty} a_n$ to exist and be (a) finite and non-zero, (b) zero, (c) infinite.
2. Are the conditions: $p_i \geqslant 0$ for $i = 1, 2, \cdots$, $p_1 > 0$ and

$$\sum_{i=1}^{k} p_i = 1$$

sufficient for $\lim_{n\to\infty} a_n$ to exist, be finite and be non-zero?

SOME SQUARE

H-230 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

(a) If 5 is a quadratic nonresidue of a prime p ($p \neq 5$), then $p \mid F_{k(p+1)}$, k a positive integer.
(b) If 5 is a quadratic residue of a prime p, then $p \mid F_{k(p-1)}$, k a positive integer.

Solution by J. L. Hunsucker, University of Georgia, Athens, Georgia.

In problem H-221 of this Journal (Vol. 2, No. 3), L. Carlitz gave the theorem:

Let p be an odd prime, $p \neq 5$. If $p \equiv 1 \pmod 4$ then $(F_{p-1}/2) \equiv 0 \pmod p$ for $(5/p) = 1$ and $(F_{p+1}/2) \equiv 0 \pmod p$ for $(5/p) = -1$; if $p \equiv 3 \pmod 4$ then $(L_{p-1}/2) \equiv 0 \pmod p$ for $(5/p) = 1$ and $(L_{p+1}/2) \equiv 0 \pmod p$ for $(5/p) = -1$.

Using the theorem that $F_n \mid F_{kn}$ in the case $p \equiv 1 \pmod 4$ and for the case $p \equiv 3 \pmod 4$, using in addition to $F_n \mid F_{kn}$, the theorem that $L_n \mid F_m$ if and only if $m = 2kn$ we see that H-230 follows immediately from H-221.

Also solved by P. Tracy and the Proposer.

RECURRENT THEME

H-231 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

1. Let $A_0 = 0$, $A_1 = 1$, and

$$\begin{cases} A_{2k+1} = A_{2k} + A_{2k-1}\,, \\ A_{2k+2} = A_{2k+1} - A_{2k}\,. \end{cases}$$

Find A_n.

2. Let $B_0 = 2$, $B_1 = 3$, and

$$\begin{cases} B_{2k+1} = B_{2k} + B_{2k-1}\,, \\ B_{2k+2} = B_{2k+1} - B_{2k}\,. \end{cases}$$

Find B_n.

Solution by Robert M. Guili, San Jose State University, San Jose, California.

1.
$$\begin{array}{ccccccccccc} \{A_i \mid i = 0, 1, 2, \cdots\} = \{ & 0, & 1, & 1, & 2, & 1, & 3, & 2, & 5, & 3, & 8, \cdots\} \\ & & (F_2) & & (F_3) & & (F_4) & & (F_5) & & (F_6)\cdots \\ & (F_0) & & (F_1) & & (F_2) & & (F_3) & & (F_4)\cdots & \end{array}$$

$$A_{2k+1} = F_{k+2}, \qquad A_{2k+2} = F_{k+1} \quad \text{for } k = 0, 1, 2, \cdots.$$

2.
$$\begin{array}{ccccccccccc} \{B_i \mid i = 0, 1, 2, \cdots\} = \{ & 2, & 3, & 1, & 4, & 3, & 7, & 4, & 11, & 7, & 18, \cdots\} \\ & & (L_2) & & (L_3) & & (L_4) & & (L_5) & & (L_6)\cdots \\ & (L_0) & & (L_1) & & (L_2) & & (L_3) & & (L_4)\cdots & \end{array}$$

$$B_{2k+1} = F_{k+2}, \qquad B_{2k+2} = F_{k+1} \quad \text{for } k = 0, 1, 2, \cdots.$$

To derive these two solutions note that by combining the two equations

$$\begin{cases} H_{2k+1} = H_{2k} + H_{2k-1} \\ H_{2k+2} = H_{2k+1} - H_{2k}\,, \end{cases}$$

we get $H_{2k+2} = H_{2k-1}$. Using this relation to replace H_{2k} in the first equation, and H_{2k+1} in the second, we get

$$\begin{cases} H_{2k+1} = H_{2k-3} + H_{2k-1} \\ H_{2k+2} = H_{2k+4} - H_{2k-2}\,. \end{cases}$$

Now let $m = 2k - 1$, and $n = 2k + 2$ for $k = 0, 1, 2, \cdots$, which yields

$$\begin{cases} H_{m+1} = H_{m-1} + H_m \\ H_{n+1} = H_{n-1} + H_n\,. \end{cases}$$

These we recognize as the generalized Fibonacci recursive relation. By applying the starting values (A_0, A_1, A_2) and (B_0, B_1, B_2) in problems 1 and 2, respectively, we get the desired result.

Also solved by P. Tracy, A. Shannon, V. E. Hoggatt, Jr., P. Bruckman, and the Proposer.

USING YOUR GENERATOR

H-232 Proposed by R. Garfield, the College of Insurance, New York, New York.

Define a sequence of polynomials $G_k(x)\ _{k=0}^{\infty}$ as follows:

$$\frac{1}{1-(x^2+1)t^2-xt^3} = \sum_{k=0}^{\infty} G_k(x)t^k .$$

1. Find a recursion formula for $G_k(x)$.
2. Find $G_k(1)$ in terms of the Fibonacci numbers.
3. Show that when $x = 1$, the sum of any 4 consecutive G numbers is a Lucas number.

Solution by G. Wulczyn, Bucknell University, Lewisburg, Pennsylvania.

SOLUTION 1.

$$\frac{1}{1-(x^2+1)t^2-xt^3} = 1+t^2(x^2+1)+t^3x+t^4(x^2+1)^2+t^5\left[\binom{2}{1}x\right]+t^6[(x+1)^3+x^2]$$
$$+\cdots+t^{2k}\left[\binom{k}{0}(x^2+1)^k+\binom{k-1}{2}(x^2+1)^{k-3}x^2+\binom{k-2}{4}(x^2+1)^{k-6}x^4+\cdots\right]$$
$$+t^{2k+1}\left[\binom{k}{1}(x^2+1)^{k-1}x+\binom{k-1}{3}(x^2+1)^{k-4}x^3+\binom{k-2}{5}(x^2+1)^{k-7}x^5+\cdots\right].$$

SOLUTION 2.

$$\frac{1}{1-2t^2-t^3} = \frac{1}{(t+1)(1-t-t^2)} = \frac{1}{t+1}+\frac{t}{1-t+t^2} = 1-t+t^2-t^3+\cdots+F_nt^{n+1}$$
$$= t^{n+1}[F_n+(-1)^{n+1}].$$

SOLUTION 3.

$$F_n+(-1)^{n+1}+F_{n+1}+(-1)^{n+2}+F_{n+2}+(-1)^{n+3}+F_{n+3}+(-1)^{n+4}$$
$$= \frac{1}{\sqrt{5}}[a^n(1+a+a+a)-b^n(1+b+b+b)] = \frac{1}{\sqrt{5}}\left[a^n\left(\frac{4+2\sqrt{5}}{2}\right)+b^n\left(\frac{4-2\sqrt{5}}{2}\right)\right]\sqrt{5}$$
$$= a^{n+3}+b^{n+3} = L_{n+3} .$$

Also solved by C. Chouteau, P. Bruckman, A. Shannon, and the Proposer.

GENERAL–IZE

H-233 Proposed by A. G. Shannon, NSW Institute of Technology, Broadway, and The University of New England, Armidale, Australia.

The notation of Carlitz* suggests the following generalization of Fibonacci numbers. Define

$$f_n^{(r)} = (a^{nk+k}-b^{nk+k})/(a^k-b^k),$$

where $k = r-1$, and a,b are the zeros of x^2-x-1, the auxiliary polynomial of the ordinary Fibonacci numbers, $f_n^{(2)}$.

Show that

(a)
$$\sum_{n=0}^{\infty} f_n^{(r)} x^n = 1/(1-(a^k+b^k)x+(a^kb^k)x^2)\,.$$

Let $f_k = (a^{k+1}-b^{k+1})/(a-b)$, and prove that

(b)
$$f_n^{(r)} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}\,.$$

(Note that when $r = 2$ (and so $k = 1$), $f_k = f_{k-1} = 1$, $f_{k-2} = 0$, and (b) reduces to the well known

$$f_n^{(2)} = \sum_{0 \le 2m \le n} \binom{n-m}{m}\,.)$$

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

We form the series

$$\sum_{n=0}^{\infty} f_n^{(r)} x^n = \sum_{n=0}^{\infty} \left(\frac{a^{nk+k}-b^{nk+k}}{a^k-b^k}\right) x^n = \frac{a^k}{a^k-b^k} \sum_{n=0}^{\infty} (a^kx)^n - \frac{b^k}{a^k-b^k} \sum_{n=0}^{\infty} (b^kx)^n$$

$$= \frac{a^k}{a^k-b^k}\cdot\frac{1}{1-a^kx} - \frac{b^k}{a^k-b^k}\cdot\frac{1}{1-b^kx} = \left\{(1-a^kx)(1-b^kx)\right\}^{-1}$$

$$= \left\{1-xL_k+(-1)^kx^2\right\}^{-1} = \left\{1-(a^k+b^k)x+(ab)^kx^2\right\}^{-1},$$

which is the result of part (a). Now consider the series $S(x)$ defined as follows:

$$S(x) = \sum_{n=0}^{\infty} x^n \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}\,;$$

then

$$S(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{n-m} x^n \binom{m}{s}\binom{n-m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^{n-m-s}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=m}^{n} x^n \binom{m}{s-m}\binom{n-m}{s-m} f_{k-1}^{2s-2m} f_{k-2}^{2m-s} f_k^{n-s}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=m}^{n} \theta(n,m,s) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\sum_{s=m}^{n} \theta(n,m,s)$$

$$= \sum_{m=0}^{\infty}\sum_{s=m}^{\infty}\sum_{n=s}^{\infty} \theta(n,m,s)$$

$$= \sum_{m,s,n=0}^{\infty} x^{n+m+s} \binom{m}{s}\binom{n+s}{s} f_{k-1}^{2s} f_{k-2}^{m-s} f_k^n$$

$$= \sum_{m,s=0}^{\infty} x^{m+s} \binom{m}{s} f_{k-1}^{2s} f_{k-2}^{m-s} \sum_{n=0}^{\infty} \binom{n+s}{n} (xf_k)^n =$$

$$= \sum_{m,s=0}^{\infty} x^{m+s}\binom{m}{s} f_{k-1}^{2s} f_{k-2}^{m-s}(1-xf_k)^{-s-1}$$

$$= (1-xf_k)^{-1}\sum_{m=0}^{\infty} x^m f_{k-2}^m \sum_{s=0}^{\infty}\binom{m}{s}\left\{\frac{xf_{k-1}^2}{f_{k-2}(1-xf_k)}\right\}^s$$

$$= (1-xf_k)^{-1}\sum_{m=0}^{\infty}(xf_{k-2})^m\left\{1+\frac{xf_{k-1}^2}{f_{k-2}(1-xf_k)}\right\}^m$$

$$= (1-xf_k)^{-1}\left\{1-xf_{k-2}-\frac{x^2f_{k-1}^2}{1-xf_k}\right\}^{-1}$$

$$= \left\{(1-xf_k)(1-xf_{k-2})-x^2f_{k-1}^2\right\}^{-1}$$

$$= \left\{1-x(f_k+f_{k-2})+x^2(f_kf_{k-2}-f_{k-1}^2)\right\}^{-1}$$

$$= \left\{1-x(F_{k+1}+F_{k-1})+x^2(F_{k+1}F_{k-1}-F_k^2)\right\}^{-1} \quad (F_k \text{ is the } k^{th} \text{ Fibonacci number})$$

$$= \left\{1-L_kx+(-1)^kx^2\right\}^{-1} = \sum_{n=0}^{\infty} f_n^{(r)}x^n, \text{ by part (a).}$$

Comparing coefficients of the power series, this establishes part (b). N.B. $F_{(n+1)k}/F_k = f_n^{(r)}$.

Also solved by the Proposer.

Editorial Note: Dale Miller's name appeared incorrectly in H-237.

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Returning to (2) above, we can generate multigrades of higher orders. (For the standard method employed, see below.[1]) I give now, as an example, a third-order multigrade:

$$A_1F_1^m + A_2F_2^m \cdots A_{n-1}F_{n-1}^m + \left(\sum_1^{n-1} A-2\right)F_n^m + A_1(2F_n-F_1)^m + A_2(2F_n-F_2)^m \cdots A_{n-1}(2F_n-F_{n-1})^m$$

$$+ \left(\sum_1^{n-1} A-2\right)F_n^m = A_1(F_n-F_1)^m + A_2(F_n-F_2)^m \cdots A_{n-1}(F_n-F_{n-1})^m + \left(\sum_1^{n-1} A-2\right)0^m$$

$$+ A_1(F_n+F_1)^m + A_2(F_n+F_2)^m \cdots A_{n-1}(F_n+F_{n-1}) + \left(\sum_1^{n-1} A-2\right)(2F_n)^m$$

(where $m = 1,2,3$).

[I have added F_n to each term in (2), and added the L.H.S. totals to the original R.H.S. and *vice versa.*] Expressed more tidily, the above becomes

$$A_1[(F_1)^m + (2F_n-F_1)^m] + A_2[(F_2)^m + (2F_n-F_2)^m] \cdots A_{n-1}[(F_{n-1})^m - (2F_n-F_{n-1})^m] + 2\left[\sum_1^{n-1} A-2\right]F_n^m$$

$$= A_1[(F_n-F_1)^m + (F_n+F_1)^m] + A_2[(F_n-F_2)^m + (F_n+F_2)^m] \cdots A_{n-1}[(F_n-F_{n-1})^m + (F_n+F_{n-1})^m]$$

$$+ \left(\sum_1^{n-1} A-2\right)[(2F_n)^m + 0^m] \qquad (\text{where } m = 1, 2, 3).$$

Again, if we add any quantity B to each term, the final 0^m terms each become B^m.

REFERENCE

1. M. Kraitchik, *Mathematical Recreations,* George Allen & Unwin, London, 1960, page 79.