SOME REMARKS ON THE PERIODICITY OF THE SEQUENCE OF FIBONACCI NUMBERS

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In the work of Wall [2], a function ϕ was defined by "$\phi(m)$ is the length of the period of the sequence of Fibonacci numbers reduced to least non-negative residues modulo m, for $m > 2$." Thus, the domain of ϕ is the set of positive integers greater than 2, and the range was shown to be a subset of the set of all even integers. Below, I determine the range of ϕ exactly. In [1] I proved the following

Theorem A. If m is an integer greater than 3 then $\phi(F_m) = 2m$ if m is even and $\phi(F_m) = 4m$ if m is odd.

Here, F_m is the m^{th} Fibonacci number, where

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1} \quad (n \geqslant 1).$$

Theorem 2 of [2] shows that the values of ϕ are completely known provided its values at all prime powers are known. But, as the table of values included in [2] shows, the values that ϕ takes at primes do not seem to follow any simple pattern. In an attempt to find more of the values of ϕ I will prove the following

Theorem B. If $m \geqslant 2$ then $\phi(F_{m-1} + F_{m+1}) = 4m$ if m is even and $\phi(F_{m-1} + F_{m+1}) = 2m$ if m is odd.

Theorems A and B have the following

Corollary. The range of ϕ is the set of all even integers greater than 4.

Proof. It is clear that we cannot have an integer n for which $\phi(n) = 2$ or $\phi(n) = 4$. Suppose that r is an even integer other than 2 or 4. If r is a multiple of 4, say $r = 4s$, then $\phi(F_{s-1} + F_{s+1}) = r$ if s is even, while $\phi(F_s) = r$ if s is odd and $s > 3$. Also $\phi(F_6) = 12$. If r is not a multiple of 4, say $r = 2s$, where s is odd and $s > 1$, then

$$\phi(F_{s-1} + F_{s+1}) = r.$$

A subsidiary result is required to prove Theorem B. In the following, the symbol $\equiv$ denotes congruence modulo $(F_{m-1} + F_{m+1})$.

Lemma. For $1 \leqslant r \leqslant m$ let $G_r = F_{m-1} + F_{m+1} - F_r$. Then

(i)
$$F_{m+r} \equiv \begin{cases} F_{m-r} & \text{if } 0 \leqslant r \leqslant m \text{ and } r \text{ is even} \\ G_{m-r} & \text{if } 1 \leqslant r \leqslant m-1 \text{ and } r \text{ is odd.} \end{cases}$$

If m is a positive even integer then

(ii)
$$F_{2m+r} \equiv G_r \quad \text{if } 0 \leqslant r \leqslant m .$$

(iii)
$$F_{3m+r} \equiv \begin{cases} G_{m-r} & \text{if } 0 \leqslant r \leqslant m \text{ and } r \text{ is even} \\ F_{m-r} & \text{if } 1 \leqslant r \leqslant m-1 \text{ and } r \text{ is odd .} \end{cases}$$

Proof. We prove these results by induction on r.

(i) The assertion here is trivially true if $r = 0$ or $r = 1$. Suppose the result is true for $r - 1$ and r. If $r + 1$ is odd then

$$\begin{aligned} F_{m+r+1} &= F_{m+r} + F_{m+r-1} \equiv F_{m-r} + G_{m-r+1} \quad \text{by hypothesis} \\ &= F_{m-1} + F_{m+1} + F_{m-r} - F_{m-r+1} \\ &= F_{m-1} + F_{m+1} - F_{m-(r+1)} = G_{m-(r+1)} . \end{aligned}$$

If $r+1$ is even then

$$\begin{aligned} F_{m+r+1} &= F_{m+r} + F_{m+r-1} \\ &\equiv G_{m-r} + F_{m-(r-1)} \quad \text{by hypothesis} \\ &= F_{m-1} + F_{m+1} + F_{m-(r+1)} \\ &\equiv F_{m-(r+1)} \quad . \end{aligned}$$

(ii) The case in which $r = 0$ follows directly from (i) with $r = m$. The result is also true for $r = 1$ because

$$\begin{aligned} F_{2m+1} &= F_{2m} + F_{2m-1} \\ &\equiv F_0 + G_{m-(m-1)} \quad \text{by (i)} \\ &= G_1 \quad . \end{aligned}$$

Suppose the result is true for $r - 1$ and r. Then

$$\begin{aligned} F_{2m+r+1} &= F_{2m+r} + F_{2m+r-1} \\ &\equiv G_r + G_{r-1} \quad \text{by hypothesis} \\ &\equiv F_{m-1} + F_{m+1} - F_{r+1} \\ &= G_{r+1} \quad . \end{aligned}$$

(iii) The case in which $r = 0$ follows directly from (ii) with $r = m$. When $r = 1$ we have

$$\begin{aligned} F_{3m+1} &= F_{3m} + F_{3m-1} \\ &\equiv G_m + G_{m-1} \quad \text{by (ii)} \\ &= F_{m-1} + 2F_{m+1} - F_m \\ &\equiv F_{m-1} \quad , \end{aligned}$$

so that the result is true for $r = 1$. Suppose it is true for $r - 1$ and r. If $r + 1$ is odd then

$$\begin{aligned} F_{3m+r+1} &= F_{3m+r} + F_{3m+r-1} \\ &\equiv G_{m-r} + F_{m-r+1} \quad \text{by hypothesis} \\ &\equiv F_{m-(r+1)} \quad , \end{aligned}$$

while if $r + 1$ is even we have

$$\begin{aligned} F_{3m+r+1} &= F_{3m+r} + F_{3m+r-1} \\ &\equiv F_{m-r} + G_{m-r+1} \\ &= G_{m-(r+1)} \quad . \end{aligned}$$

This finishes the proof of the Lemma.

We may now prove Theorem B by noticing that if m is even then the sequence of Fibonacci numbers reduced modulo $(F_{m-1} + F_{m+1})$ consists of repetitions of the numbers

$$F_0, F_1, \cdots, F_m, F_{m+1}, F_{m-2}, G_{m-3}, F_{m-4}, G_{m-5}, \cdots, F_2, G_1, 0,$$

$$G_1, G_2, \cdots, G_{m-1}, G_m, F_{m-1}, G_{m-2}, F_{m-3}, G_{m-4}, \cdots, G_2, F_1,$$

while if m is odd we obtain

$$F_0, F_1, \cdots, F_m, F_{m+1}, F_{m-2}, G_{m-3}, F_{m-4}, G_{m-5}, \cdots, G_2, F_1 \ .$$

Thus, counting, and noticing that $G_1 \neq F_1$, we obtain the required results.

Using Theorem A, it may be shown that if $m > 4$ then

$$\phi(F_{m-1} + F_{m+1}) = \tfrac{1}{2}(\phi(F_{m-1}) + \phi(F_{m+1})).$$

I conclude by conjecturing that if k is a positive integer with $m - k > 3$ then

$$\phi(F_{m-k} + F_{m+k}) = \frac{k}{2}\,(\phi(F_{m-k}) + \phi(F_{m+k})).$$

REFERENCES

1. T. E. Stanley, "A Note on the Sequence of Fibonacci Numbers," *Math. Mag.*, 44, No. 1 (1971), pp. 19–22.
2. D. D. Wall, "Fibonacci Series Modulo *m*," *Amer. Math. Monthly*, 67 (1960), pp. 525–532.

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PARITY TRIANGLES OF PASCAL'S TRIANGLE

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In the Pascal's triangle of binomial coefficients, $\binom{n}{r}$, let every odd number be represented by an asterisk, "*," and every even number by a cross, "†." Then we discover another diagram which is quite interesting.

Every nine (odd) numbers form a triangle having exactly one (odd) even number in its interior (odd!). Thus we shall designate it as an Odd-triangle.

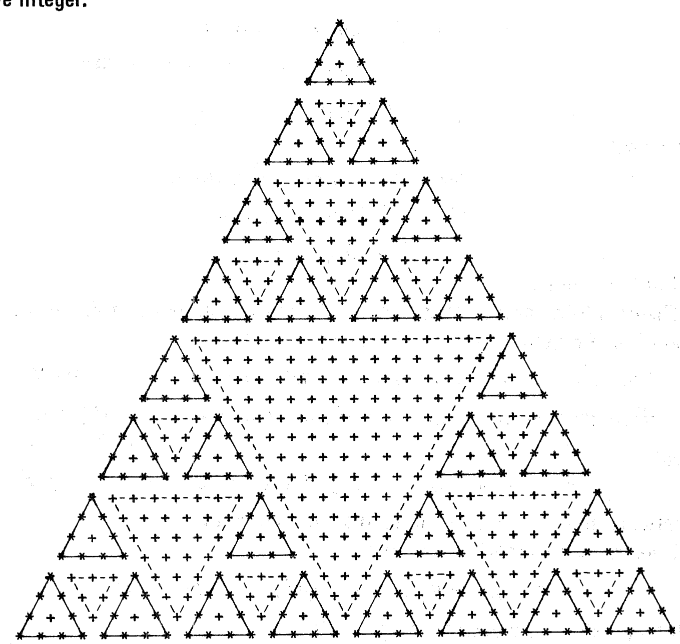
The even numbers also form triangles whose sizes vary but each of these triangles contains an even number of crosses. This set of triangles is called Even-triangles.

The present diagram $(n = 31)$ can be easily extended along the outermost apex of Pascal's triangle. Some partial observations are:

(a) If $n = 2^i - 1$ and $0 \leqslant r \leqslant 2^i - 1$, then $\binom{n}{r}$ is odd,

(b) If $n = 2^i$ and $1 \leqslant r \leqslant 2^i - 1$, then $\binom{n}{r}$ is even,

where i is a nonnegative integer.



Parity Triangles of $\binom{n}{r}$

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