BODE'S RULE AND FOLDED SEQUENCES

W. ELLIOTT GREIG
West Virginia University, Morgantown, West Virginia 26506

THE THEORY

I have discovered a new and most interesting variation on recursions during my work on Bode's rule. Fibonacci-Lucas (F-L) sequences, by definition, satisfy $H_{k+1} = H_k + H_{k-1}$. A folded or crimped-in-upon-itself infinite sequence is N-cyclic and breaks the F-L rule only *once* per cycle, *i.e.,*

(1) $$G_{k+1} = G_k + G_{k-1} \quad \text{except that} \quad \{G_N\}_{jN} = \{G_N\}_0 \quad \text{for all } j,\ N.$$

As an example the $N = 3$ case is $\{G_3\} = \cdots 2, 0, 2, 2, 0, 2, 2, 0, \cdots$ in which $\{G_3\}_0 = \{G_3\}_3$. Application of (1) $N - 1$ times gives

(2) $$G_0F_{N-1} + G_1F_N = G_0\,,$$

where $\{F\}$ is Fibonacci's sequence. This determines the sequences in Table 1. The partial sum of a F-L sequence is $(F_{n+2} - F_{m+1})$ for all $m,n,$ where F_m and F_n are the first and last terms. Using this the sum over one cycle of a folded F-L sequence gives $G_{N-1} + G_N - G_1$. Since (1) and (2) give $G_k = G_{jN+k}$ for all integers j,k we have

(3) $$\sum_{k=0}^{N-1} G_k = G_{-1} + G_0 - G_1\,.$$

An easy way to generate folded F-L sequences follows. From (2), $\{G_N\}_0$ must equal or be a multiple of F_N to avoid a fractional $\{G_N\}_1$. Let $\{G_N\}_0 = F_0 + F_N$ then (2) gives $\{G_N\}_1 = F_1 - F_{n-1}$. Thus every $\{G_N\}$ is simply the sum of a positive and negative Fibonacci sequence. We have

(4) $$\{G_N\}_k = F_k + (-1)^{N+1}F_{k-N}\,, \qquad 0 \leqslant k \leqslant N$$

and using the "skew symmetric" fact that

(5) $$F_{-k} = (-1)^{k+1}F_k$$

gives the simpler expression

(6) $$\{G_N\}_k = F_k + (-1)^kF_{N-k}\,, \qquad 0 \leqslant k \leqslant N.$$

One can also define negative folded F-L sequences which are finite and of length $N + 1$. Their definition is

(7) $$\{G_{-N}\}_k = F_k + (-1)^NF_{k-N}\,, \qquad 0 \leqslant k \leqslant N.$$

An example is $\{G_{-5}\} = -5, 4, -1, 3, 2, 5$. Substitution of (4) or (6) into (3) permits an explicit sum formula:

(8) $$\sum_{k=0}^{N-1} \{G_N\}_k = L_N + (-1)^{N+1} - 1.$$

When mod $(N,4) = 0$ then

$$-G_1/G_0 = (F_{N-1} - 1)/F_N = L_{m-1}/L_m\,,$$

where $N = 2m$. The proof consists of crossmultiplying and inserting identity (7) of Hoggatt [1] which is true for all integers m. This reduces to a special case of identity (21) in the list [1]. Thus mod $(N,4) = 0$ gives folded Lucas sequences. Similarly when mod $(N,4) = 2$ then $-G_1/G_0 = F_{m-1}/F_m$ giving folded Fibonacci sequences. The proof is identical and ends with a special case of identity (23) in his list [1]. But the interesting cases are mod $(N,4) = 1$ or 3.

Table 1
Folded Fibonacci-Lucas Sequences

N	ONE CYCLE OF SEQUENCE	SUM
2	1 0	1
4	3 −1 2 1	5
6	8 −4 4 0 4 4	16
8	21 −12 9 −3 6 3 9 12	45
10	55 −33 22 −11 11 0 11 11 22 33	121
3	2 0 2	4
5	5 −2 3 1 4	11
7	13 −7 6 −1 5 4 9	29
9	34 −20 14 −6 8 2 10 12 22	76
11	89 −54 35 −19 16 −3 13 10 23 33 56	199
13	233 −143 90 −53 37 −16 21 5 26 31 57 88 145	521
15	··· −142 92 −50 42 −8 34 26 ···	1364
33	··· −1974;2584;610;3194;3804;6998;10802···	
∞	$\cdots -\sqrt{5}-1; \sqrt{5}+2;\ 1; \sqrt{5}+3; \sqrt{5}+4 \cdots$	

The reciprocal periods of planets and satellites are given by alternate members of an odd N-folded sequence. Their properties are studied best by placing the origin in the middle hence I define a half-integer subscript i given by $2k = 2i + N$.

Theorem: $\{G_N\}_{i+1}/\{G_N\}_i$ approaches a limiting value for all i as $N \to \infty$ for mod $(N,4) = 1$ and another for mod $(N,4) = 3$.

Proof: It is sufficient to prove this for one value of i whence it is true for all i by (1) aside from a constant factor which is of no interest. Write $h = \frac{1}{2}$ for typographical ease. I also define even integer $m = (N \pm 1)/2$ when mod $(N,4) = 3$ or 1, respectively. Then by (4) the middle pair for mod $(N,4) = 1$ is

$$\{G_N\}_h/\{G_N\}_{-h} = (F_{m+1} + F_{-m})/(F_m + F_{-m-1}) = F_{m-1}/F_{m+2} \to 1/\alpha^3 \quad \text{as} \quad N \to \infty$$

via Binet's expression since $\beta^N \to 0$ as $N \to \infty$, where $\alpha, \beta = (1 \pm \sqrt{5})/2$, respectively. For mod $(N,4) = 3$ we have:

$$\{G_N\}_h/\{G_N\}_{-h} = -F_{m+1}/F_{m-2} \to -\alpha^3 \quad \text{as} \quad N \to \infty$$

for the same reason. These initial ratios $\pm\alpha^{\mp 3}$ define $\{S\}$ and $\{S^*\}$ and apply to any star (planet) with an infinite number of planets (satellites). When mod $(N,4) = 1$ let $S_{-h} = 2 + \sqrt{5}$ and $S_h = 1$ as in Table 1. Then (1) gives us

(9a) $$S_i = F_{i+h}S_h + F_{i-h}S_{-h}$$

for all positive or negative half-integers i. Similarly when mod $(N,4) = 3$ let $S^*_{-h} = -1$ and $S^*_h = 2 + \sqrt{5}$ then

(9b) $$S^*_i = F_{i+h}S^*_h + F_{i-h}S^*_{-h}\ .$$

Substitution of (5) into (9a,b) proves the equivalence of S and S^* but for signs *i.e.,*

(10) $$S^*_i = (-1)^{i-h}S_{-i} \quad \text{or} \quad S^*_{-i} = (-1)^{i+h}S_i\ .$$

Use of $F_{i+h} + 2F_{i-h} = L_{i+h}$ in (9a,b) gives the elegant relations

(11) $$S_i = L_{i+h} + \sqrt{5}F_{i-h} \quad \text{and} \quad S^*_i = L_{i-h} + \sqrt{5}F_{i+h}$$

(11a) $$S_i = s_0(\sqrt{5}F_{i+h} - L_{i-h}) \quad \text{and} \quad S^*_i = s_0(L_{i+h} - \sqrt{5}F_{i-h})$$

which via Binet's theorem become

(12a) $$S_i = (a^i + (-1)^{i+h} a^{-i})\sqrt{s_0}$$

(12b) $$S_i^* = (a^i + (-1)^{i-h} a^{-i})\sqrt{s_0}$$

which immediately give (10) again and where $\sqrt{s_0} = (a^h + a^{-h}) = 2.058171 = 1/0.485868$.

The Lucas complement of any two-point sequence is defined by the two apart sum operator $\Sigma^\dagger$, namely

(13) $$\Sigma^\dagger W_n = (W_{n+1} + W_{n-1})/d\ ,$$

where d is the difference in the roots of the recursion's characteristic equation [2] and $d = \sqrt{5}$ for F-L sequences. It is known that $\Sigma^\dagger\Sigma^\dagger \equiv I$, the identity operator. We come now to the strongest property of $\{S\}$ and $\{S^*\}$. Aside from signs they are their *own* complements! The fact that this property is not true of the Fibonacci and Lucas sequences themselves indicates the greater importance of $\{S\}$ and its approximation $\{G_N\}$. After all $\{G_N\}$ is a generalization of $\{F\}$ and $\{L\}$. Applying $\Sigma^\dagger$ to the elegant (11) immediately gives

(14) $$\Sigma^\dagger S_i = S_i^* = (-1)^{i-h} S_{-i}$$

since

$$\sqrt{5}\,\Sigma^\dagger F_n = L_n \qquad \text{and} \qquad \Sigma^\dagger L_n = \sqrt{5}\,F_n\,.$$

Alternatively given (14) we can ask what the ratio S_h/S_{-h} in (9) must be. One obtains

$$S_{-h}^2 - 4S_{-h}S_h - S_h^2 = 0.$$

2. THE OBSERVATIONS

Several facts of satellites (planets) need to be explained. They can be remembered using the vowel mnemonic, $aei\underline{\omega}\omega\epsilon A$. They are: (i) rule(s) for the major semi-axes of the orbits, (ii) their near zero eccentricities, (iii) $\sin i \approx 0$, *i.e.,* their orbital inclinations are nearly 0 or 180° for outer satellites, (iv) their spins are almost all counterclockwise (ccw) with a preference for $23° < \underline{\omega} < 29°$ where the sun and Jupiter are prominent exceptions, (v) their spins satisfy the narrow range $6 < P < 25$ hr unless tidally disturbed, (vi) the sun's obliquity $\epsilon = 7°$ hence the sun's equator does not lie in the invariable plane, (vii) the sun's Angular momentum is very small (it rotates in ≈ 30 day). I add (viii) that each satellite system has one or two satellites much more massive than the others. The massive satellites are called secondaries and all others are tertiaries. Thus Saturn's and Jupiter's secondaries are Titan+Hyperion and Galilei's quadruplet, respectively. The non-zero tilt of most of their axes suggests that the torque that each exerts on the other causing precession may be important. The ideal tilt is then 45°.

I envisage that the sun's family began with the sun and Jupiter (+ ?) Saturn from a contracting cloud and that all planetary and satellite systems start as binary systems, *i.e.,* a primary + secondary(ies). All other bodies, tertiaries, were subsequently formed by accretion. The sun's nebula would have dispersed early due to radiation pressure and infalling due to the Poynting-Robertson effect. Many planets and satellites should have formed from the nebula left around Jupiter. Binaries enable the capture of tertiary bodies. A single primary cannot capture a tertiary body whose orbit must *ab initio* be an ellipse or hyperbola. Outer satellites, those beyond the secondaries, act as if the secondaries were part of the primary. When the maximum elongation angle of the secondaries is very small they act as a point source. The number of major planets makes $N = 33$. Although N may be slightly different for the satellite systems $\{G_N\} \to \{S\}$ rapidly and for $N \geqslant 13$ the discrepancies are < one percent as Table 1 shows.

In Table 2, major bodies are capitalized. Also pons, faye, neujmin and hungaria refer to groups of comets at $61 < P < 77$ yr, $6.3 < P < 7.9$ yr, $P = 18$ yr, and a group of small planetoids $2.5 < P < 3.0$ yr named after the first discovered [3] 434 Hungaria ($P = 991$ da). There is a void in the planetoid distribution [4, p. 169] separating these from the normal asteroids indicated by a typical member Astrea. See also [10, 11]. Note that satellites of Saturn and Jupiter are included. The accuracy is very high. Discrepancies are never more than 2.5 percent except Jupiter (6%) and Galilei's quadruplet (10%) both of which are secondaries for which the rule is not intended. The observed Iapetus/Phoebe ratio (outer satellites of Saturn) is 6.938. The predicted ratio is very nearly $(76 + 21\sqrt{5})/(11 + 3\sqrt{5})$

= 75004/10802 ≈ 6.9435. This is amazing agreement, an error of 0.0008 parts! The observed and predicted Saturn/Uranus ratios are 2.852 and 10802/3804 = 2.840, an error of 0.004. The Direct/Retrograde satellites of Jupiter give 2.835. Again excellent agreement. The agreement for Venus/Earth is similar. The data in [5] give JXII/Retrograde = 1.18 compared with a predicted 1.20. Planet X was predicted [6] from perturbations of Halley's comet. I find its period to be 521 to 524 yr. Bailey [7] proposed that the moon was once between Venus and Mercury.

Table 2
Reciprocal Periods

Sun	pons	Nep		X		Uran		Satur	neujm	Jup	faye
3524578, ···	4558,	−1974,	2584,	610,	3194,	3804,	6998,	10802,	17800,	28602,	46402
	JXII			*		Retr		Direct			
								Phoebe			

Astrea	Hungar	Mars	earth	Venus	luna?	Mercury		Sun	
075004,	121406,	196410,	317816,	514226,	832042,	1346268,	2178310,	3524578,	(5702888)
Iapetus			T+H		Galilei	Rhea	dione	Tethys	Almathea

The predicted effective solar rotation period is 32.8 day. If all planetary (and satellite) systems have about the same number of bodies and if these are tied to the primary's rotation then stars rotating much faster than the sun will have their planets too close to permit life. This would be true of white stars (earlier than type F5) whose rotation period is about 0.01 of the sun's. The theory predicts Mercury's period to be 86 ± 0.2 day. Hence some mechanism decreased its orbital energy and increased its orbital angular momentum. Furthermore the planetary rotations seem to be quantized near 1.14, 0.70, 0.43, 0.27 (asteroids), 0.17, 0.10 (solar grazing body) day. Although Folded sequences have made excellent predictions far more accurate than any previous work the sequences for Jupiter's and Saturn's satellites can be modified to include the sun's motion around the planet. For Saturn's satellites the sequence: 0.093, 0.636, 1.815, 4.809, 12.612 ··· is equally good. Similarly a Jovian sequence: 0.23, 1.35, 3.82, 10.11 ··· is a good predictor. The units are $(\text{kiloday})^{-1}$. The first term of each sequence is the motion of planet and sun around each other and is already determined by the sequence for the planets [8, 11].

Alternate F-L members approach the limit $\xi = (3+\sqrt{5})/2$. The limiting distance ratio, d, is given by Kepler's III law: $a^3 = p^2$. Hence $d = 1.899547627$. Planets and satellites were accreted from grain orbits of maximum eccentricity e thus $(1-e)/(1+e) = 1/d$. This function occurs often in science and deserves its own name. I define

$$oin(x,p) = (1-x)/(1+px)$$

because it is its own inverse, *i.e.*, if $y = oin(x,p)$ then $x = oin(y,p)$, where p is a parameter. This gives $e = 0.3102$.

3. INNER SATELLITES

Once again the god of time Chronos or Saturn holds the secret. Table 3 gives the reciprocal periods, Ω,

Table 3
Inner Saturnian Satellites

Rhea		Dione		Tethys		M+2E		Janus		Rotn?			Cassin	Ω_g?
221.4		365.4		529.7		838.0		1310.6		2091.5		3345.0		5379.4
	144.0		164.3		308.3		472.6		780.9		1253.5		2034.4	
123.7		20.3		144.0		164.3		308.3		472.6		780.9		1253.5

in $(\text{kiloday})^{-1}$. The errors for Rhea, Dione and Tethys are each 0.01 percent! The first differences are the synodic frequencies and they are a F-L sequence! Cassini's division falls on one of these values. Slightly different sequences occur in [9]. I must define two new operators. They are a forward knight operator $K \equiv \Delta + \Delta^2$ and a backward knight operator $N \equiv \nabla - \nabla^2$ by analogy with the chess piece. More generally:

$$K^p \equiv \Delta^p \sum_{x=0}^{p} \binom{p}{x} \Delta^x \tag{15}$$

For any F-L sequence $K^p F_n = F_n$ for all n and all integers $p \geqslant 0$. Carried leftwards Table 3 predicts the period of a grazing satellite to be 0.155 day. Table 3 is a shifted F-L sequence. It satisfies

$$K\Omega_n + \Omega^\circ = \Omega_n, \tag{16}$$

where Ω° = 0,0571 inverse days. But Ω° is very nearly the mean reciprocal period of massive Titan and Hyperion! For the terrestrial planets the situation is almost as good. Errors are negligible but for Mars. Here we have (16) with

Table 4
Terrestrial Planets in $(\text{Kiloday})^{-1}$

Hungar		(Mars)		Earth		Venus		luna?		Mercury		(Vulcan)		
1.026		1.851		2.738		4.450		7.049		11.360		18.270		29.49
	0.825		0.887		1.712		2.599		4.311		6.910		11.221	
		0.062		0.825		0.887		1.712		2.599		4.311		

Ω° = 0.000 139 invday. This compares well with the synodic frequency between Jupiter and Saturn, 0.000 137 9. Hence the frequencies of inner planets are increased by the frequency of Jupiter and Saturn conjunctions. Carried leftwards Table 4 suggests a solar rotation of 25 day. The Martian error can [9] be removed by writing Ω° = –.000 256 at a small expense to Mercury. The inner Uranian triplet satisfies

$$\delta^2\Omega = \nabla\Omega \qquad \text{and} \qquad \Delta\Omega_n + \Omega^\circ = \Omega_n ,$$

where Ω° = 0.0858 invday and δ is the central difference operator.

COLOPHON

Johannes Kepler's Zeroth Law appeared in the year F_{17}. It was the first cosmological attempt and states that planets orbit in spheres which in- and circum-scribe the F_5 perfect solids arranged in the order 2, 8, –8, 0, –2 of faces minus vertices—all but one, members of $\{F\}$. It is to his faith in pure mathematics that I am indebted. F_{14} years later I found that the universal answer is $(\delta^2 - I)\Omega \to 0$. Another genius, J. C. Maxwell, also began his life [11, p. 93] studying the perfect solids and provided us with an elegant derivation of kinetic theory to which I am also indebted. Pussy willow leaves [1] and Houseleek petals display the 5/13 arrangement. And in haiku let us say:

Nature numbers hides
In shells, petals, moons to find
Is to hear with her

and in tanka style:

Each conjunction that I see
So real that
'junctions too must fit the rule
For are not 'junctions real
And earth's motion their conjunctions.

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DIOPHANTINE REPRESENTATION OF THE LUCAS NUMBERS

JAMES P. JONES
University of Calgary, Calgary, Alberta, Canada

The Lucas numbers, 1, 3, 4, 7, 11, 18, 29, ⋯, are defined recursively by the equations

$$L_1 = 1, \qquad L_2 = 3 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n .$$

We shall show that the Lucas numbers may be defined by a particularly simple Diophantine equation and thus exhibit them as the positive numbers in the range of a very simple polynomial of the 9th degree.

Our results are based upon the following identity

$$L_{n+1}^2 - L_{n+1}L_n - L_n^2 = 5(-1)^{n+1} . \tag{1}$$

This identity (cf. [1] p. 2 No. 6) actually *defines* the Lucas numbers in the following sense.

Theorem 1. For any positive integer y, in order that y be a Lucas number, it is necessary and sufficient that there exist a positive number x such that

$$y^2 - yx - x^2 = \pm 5 . \tag{2}$$

Proof. The Proof is virtually identical to that of the analogous result for Fibonacci numbers proved in [2].

Theorem 2. The set of all Lucas numbers is identical with the position values of the polynomial

$$y(1 - ((y^2 - yx - x^2)^2 - 25)^2) \tag{3}$$

as the variables x and y range over the positive integers.

Proof. We have only to observe that the right factor of (3) cannot be positive unless equation (2) holds. Here we are using an idea of Putnam [3].

It will be seen that the polynomial (3) also gives certain negative values. This is unavoidable. It is easy to prove that a polynomial which takes *only* Lucas number values must be constant (cf. [2] Theorem 3).

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