IN-WINDING SPIRALS

VERNER E. HOGGATT, JR.
San Jose State University, San Jose, California 95192
and
KRISHNASWAMI ALLADI
Vivekananda College, Madras 600004, India

1. INTRODUCTION

In [1] Holden discusses the system of outwinding squares met in the geometric proof of

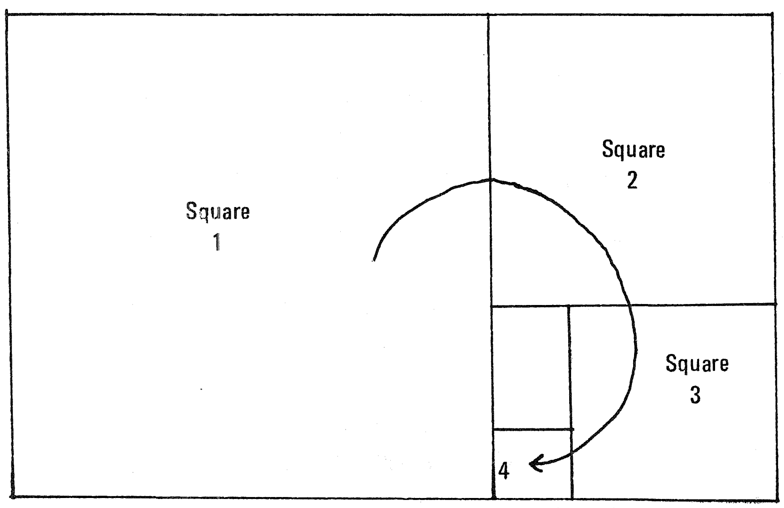
$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1} .$$

In fact, it is shown there that the centers of the squares lie on two orthogonal lines with slopes $-1/3$ and 3, respectively. This was the original problem. Further, if one lets $u_1 = 1$, $u_2 = p$, and $u_{n+2} = u_{n+1} + u_n$, one obtains the generalized Fibonacci sequence. Here the tiling is not made up of squares but contains one 1 by $(p-1)$ rectangle but the whirling squares still have their centers on two orthogonal straight lines.

It is the purpose of this paper to extend the results to in-winding systems of squares and rectangles. For background and generalizations, see Hoggatt and Alladi [2].

2. THE CLASSIC EXAMPLE

The Golden Section Rectangle yields one beautiful example of in-winding spirals of squares. We start with a rectangle such that if one cuts a square from it, then the remaining rectangle is similar to the original one. The ratio of length to width of this rectangle is $\alpha = (1 + \sqrt{5})/2$.



We now repeat the cutting off of a square from the second rectangle, then a square from the third rectangle, and so on for n steps. This will leave some 1 by $(p-1)$ rectangle in the middle of the system of squares. One immediately notices that if the rectangle is $1 \times (p-1)$, then the n^{th} square was $p \times p$ and reversing the construction you are indeed adding squares on such that the sides form a generalized Fibonacci sequence. That is, the resulting squares have their centers on two mutually perpendicular straight lines. Now, since the out-winding squares from the $1 \times (p-1)$

rectangles have centers on two mutually orthogonal lines for one n the same pair of lines hold for all n. In other words, the nested set of rectangles converges onto a point. In the case of the Golden Section rectangle the sequence of corners of the rectangles lie on two mutually orthogonal lines and further the common point of intersection of this pair of lines coincides with that of the pair of lines determined by the centers of the in-winding squares.

Suppose we let $f_0 = p$ and $f_1 = 1$, $f_2 = f_0 - f_1 = p - 1$, $f_3 = 1 - (p-1) = 2 - p$, $f_4 = (p-1) - (2-p) = 2p - 3$, $f_5 = (2-p) - (2p-3) = 5 - 3p$, $\cdots$, so that in general one gets

$$f_n = (F_{n+1} - pF_n)(-1)^n ,$$

where F_n is the n^{th} Fibonacci number. In the event that $p = \alpha$, then

$$\lim_{n \to \infty} f_n = \lim_{n \to \infty} F_n \left(\frac{F_{n+1}}{F_n} - \alpha \right) = 0.$$

This suggests that although we know

$$\lim_{n \to \infty} \left(\frac{F_{n+1}}{F_n} - \alpha \right) = 0$$

in reality

$$\lim_{n \to \infty} F_n \left(\frac{F_{n+1}}{F_n} - \alpha \right) = 0.$$

To see this, we look at

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} - \alpha = \frac{\alpha^{n+1} - \beta^{n+1} - \alpha^{n+1} + \alpha\beta^n}{\alpha^n - \beta^n} = \frac{-\beta^n(\alpha - \beta)}{\alpha^n - \beta^n} .$$

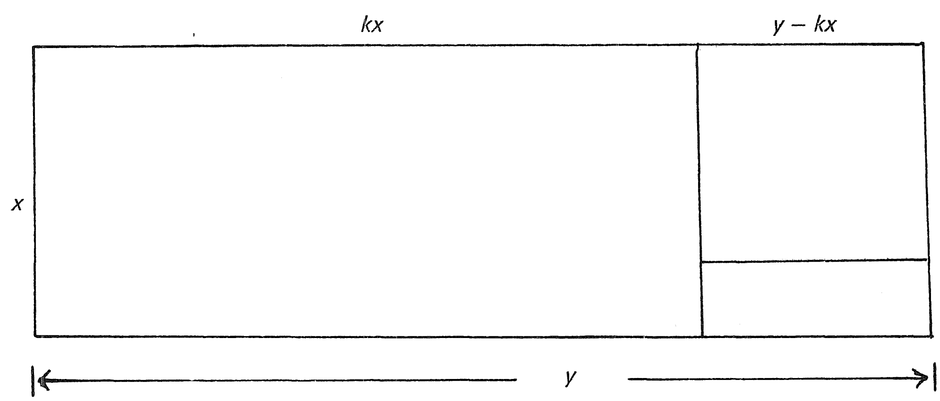
Thus,

$$F_n \left(\frac{F_{n+1}}{F_n} - \alpha \right) = -\beta^n \to 0.$$

This seems to indicate that unless $\alpha = p$, the process will not converge for squares.

3. A GENERALIZATION: THE SILVER RECTANGLE

Suppose we cut off k squares from the rectangle and then want the remaining rectangle to be similar to the original.



$$\frac{y}{x} = \frac{x}{y - kx}$$

$$y^2 - kxy - x^2 = 0$$

$$\left(\frac{y}{x}\right)^2 - k\left(\frac{y}{x}\right) - 1 = 0$$

$$\frac{y}{x} = \frac{k \pm \sqrt{k^2 + 4}}{2}$$

Since we wish $y/x > 0$, then $\alpha = (k + \sqrt{k^2+4})/2$ is selected. In reality, this leads naturally to the Fibonacci polynomials. Suppose again we start out with $f_0 = p$ and $f_1 = 1$, $f_2 = p - k$,

$$f_3 = 1 - k(p-k) = k^2 - kp + 1 = (k^2+1) - pk$$

$$f_4 = (p-k) - k(k^2 - kp + 1) = (-k^3 - 2k) + p(k^2+1) = -u_4(k) + pu_3(k)$$

$$f_n = (-1)^n [u_{n+1}(k) - pu_n(k)],$$

where $u_n(k)$ is the n^{th} Fibonacci polynomial. Once again $\lim_{n\to\infty} f_n$ does not exist unless

$$p = (k + \sqrt{k^2+4})/2;$$

then

$$f_n = (-1)^n u_n(k)\left(\frac{u_{n+1}(k)}{u_n(k)} - p\right).$$

$$\lim_{n\to\infty} f_n = 0$$

as before. When $k = 1(un(1) = F_n)$ so that unless $p = \alpha$, then

$$f_n = (-1)^n [u_{n+1}(k) - \alpha u_n(k) - (p-\alpha)u_n(k)] = (-1)\cdot 1 + (-1)^n(\alpha - p)u_n(k)$$

which diverges since $\lim_{n\to\infty} u_n(k) \to \infty$ for each $k > 0$.

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H. W. Gould
Department of Mathematics,
West Virginia University
Morgantown, W. Va. 26506

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