ON THE INFINITE MULTINOMIAL EXPANSION[1]

DAVID LEE HILLIKER
The Cleveland State University, Cleveland, Ohio 44115

Abel, [1], about 150 years ago gave the first proof of the Binomial Theorem for the case of an arbitrary complex exponent. From Abel's result one can deduce various versions of the Multinomial Expansion. In this note we shall derive one such form.

Let $n, a_1, a_r, \cdots, a_r$ be complex numbers with n not equal to a non-negative integer. *If the inequalities*

$$|a_j| < |a_1 + a_2 + \cdots + a_{j-1}|, \tag{1}$$

for $j = 2, 3, \cdots, r$, *all hold, then the following Multinomial Expansion holds:*

$$\left(\sum_{i=0}^{n} a_i\right)^n = \sum \frac{n(n-1)\cdots(n-n_1-n_2-\cdots-n_{r-1}+1)}{n_1!\,n_2!\cdots n_{r-1}!}\; a_r^{n_1} a_{r-1}^{n_2} \cdots a_2^{n_{r-1}} a_1^{n-n_1-n_2-\ldots-n_{r-1}}, \tag{2}$$

where the summation is an iterated summation taken under all $n_i \geqslant 0$, *where* i *first takes on the value* $r-1$, *then* $r-2$, *and so on until the last value, 1, is taken on.*

We first establish the following *triple summation expansion:*

$$\left(\sum_{i=1}^{r} a_i\right)^n = \sum_{j=2}^{r}\sum_{k=1}^{\infty}\binom{n}{k} a_j^k \left(\sum_{\ell=1}^{j-1} a_\ell\right)^{n-k} + a_1^n \tag{3}$$

if the inequalities (1) all hold. Here we use the usual convention that $\binom{n}{k} = 0$ when n is a positive integer and $k > n$. Formula (3) is of interest in its own right. This author has found it, as well as Formula (7), to be of use in the representation of integers in specialized arithmetical systems, such as the binary system.

Indeed, let $z_1 = 0$ and

$$z_j = \sum_{\ell=1}^{j-1} a_\ell$$

for $j \geqslant 2$, so that the right side of (3) becomes, by (1),

$$\sum_{j=1}^{r} \left((z_j + a_j)^n - z_j^n\right) = \sum_{j=1}^{r} \left(z_{j+1}^n - z_j^n\right) = z_{r+1}^n,$$

which is precisely the left side of (3).

Since $n - k \neq 0$, we can apply Formula (3) to the summation under ℓ on the right side of (3). This iterative process can be continued. After m iterations of Formula (3), $m \geqslant 0$ and not too large, we obtain

[1] Received by the editors in July, 1973.

$$(4)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2m}}{\ell_{2m+2}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2m+1}}^{\ell_{2m+2}}\left(\sum_{\ell=1}^{\ell_{2m+1}-1} a_\ell\right)^{n-\ell_2-\cdots-\ell_{2m+2}}$$

$$+\sum_{k=1}^{m}\sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}} a_1^{n-\ell_2-\cdots-\ell_{2k}} + a_1^n.$$

Here the indices vary over

$$(5)\qquad \begin{cases} 2 \leqslant \ell_1 \leqslant r, \\ 2 \leqslant \ell_{2i+1} \leqslant \ell_{2i-1}-1, & \text{for } 1 \leqslant i \leqslant m, \\ 1 \leqslant \ell_{2i+2} < \infty, & \text{for } 0 \leqslant i \leqslant m. \end{cases}$$

The only restriction on m is that $m \leqslant r-2$, so that the first two inequalities in (5) are possible.

We let $m = r-2$, for $r \geqslant 2$. Then, by (5), $\ell_{2r-3} = 2$, so that Formula (4) becomes

$$(6)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2r-4}}{\ell_{2r-2}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2r-3}}^{\ell_{2r-2}} a_1^{n-\ell_2-\cdots-\ell_{2r-2}}$$

$$+\sum_{k=1}^{r-2}\sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}} a_1^{n-\ell_2-\cdots-\ell_{2k}} + a_1^n$$

$$= \sum_{k=1}^{r-1}\sum \binom{n}{\ell_2}\binom{n-\ell_2}{\ell_4}\cdots\binom{n-\ell_2-\cdots-\ell_{2k-2}}{\ell_{2k}} a_{\ell_1}^{\ell_2} a_{\ell_3}^{\ell_4}\cdots a_{\ell_{2k-1}}^{\ell_{2k}} a_1^{n-\ell_2-\cdots-\ell_{2k}} + a_1^n .$$

We now extend the range of ℓ_{2i}, for $1 \leqslant i \leqslant r-1$, to include 0. Then, the summation under k reduces to a single term $k = r-1$; and, by (5), the subscripts are uniquely determined:

$$\ell_1 = r, \qquad \ell_3 = r-1, \quad \cdots, \quad \ell_{2r-3} = 2.$$

It now follows from (6) that

$$\left(\sum_{i=1}^{r} a_i\right)^n = \sum \frac{n(n-1)\cdots(n-\ell_2-\cdots-\ell_{2r-2}+1)}{\ell_2!\,\ell_4!\cdots\ell_{2r-2}!}\, a_r^{\ell_2} a_{r-1}^{\ell_4}\cdots a_2^{\ell_{2r-2}} a_1^{n-\ell_2-\cdots-\ell_{2r-2}}$$

this result being valid for all $r \geqslant 1$. Here, we are employing the usual convention that the empty sum is 0 and the empty product is 1.

The Multinomial Expansion (2), subject to the restrictions (1), now follows with a change of notation.

Another version of the Multinomial Theorem is

$$(7)\qquad \left(\sum_{i=1}^{r} a_i\right)^n = (-1)^r \sum_{j=2}^{r} (-1)^j \left[\sum_{k=1}^{\infty}\binom{n}{k} a_j^k \left(\sum_{\ell=1}^{j-1} a_\ell\right)^{n-k} + 2\left(\sum_{\ell=1}^{j-1} a_\ell\right)^n\right] + a_1^n ;$$

valid under the conditions (1).

A good source for the Binomial Theorem and the Multinomial Theorem is Chrystal's *Algebra* [2], Volumes I and II. Our sequence of expository papers on the Binomial Theorem, the Multinomial Theorem, and various Multinomial Expansions (Hilliker [3], [4], [5], [6] and the present paper) will continue (Hilliker [7], [8]).

ADDENDUM. Here, as usual, z^n is defined to be that branch of the function $f(x) = e^{n \log z}$ defined over the complex z-plane with the nonpositive real axis included, and with $f(1) = 1$. That is, the logarithmic function is given by $\log z = \log|z| + i \arg z$ with $|\arg z| < \pi$. Our inequalities (1) imply that the quantities $a_1 + a_2 + \cdots + a_j$, for $1 \leqslant j \leqslant r$, are not 0. We shall need to assume that they are not negative real numbers. When n is a (negative) integer these restrictions which guarantee single-valuedness, may, naturally, be ignored. For more on this, and also for a development of the Binomial Theorem, that is, the Maclaurin expansion

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

for n and z complex and with $|z| < 1$, see Markushevich [9], I.

REFERENCES

1. Niels Henrick Abel, *Crelle's Journal,* I, 1826, p. 311. The proof for complex exponents also appears in Abel's *Oeuvres,* I, p. 219, Christiania, 1881.
2. G. Chrystal, *Textbook of Algebra,* Vols. I and II, Chelsea, New York, 1964. This is a reprint of works of 1886 and 1889. Also reprinted by Dover, New York, 1961.
3. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton, in regard to Newton's Discovery of the Binomial Theorem." First part, "Contributions of Pascal," *The Mathematics Student,* Vol. XL, No. 1 (1972), pp. 65–71.
4. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in regard to Newton's Discovery of the Binomial Theorem," second part, "Contributions to Archimedes," *The Mathematics Student,* Vol. XL, No. 4 (1973), 4 pp.
5. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in Regard to Newton's Discovery of the Binomial Theorem," third part, "Contributions of Cavalieri," *The Mathematics Student,* Vol. XLI, No. 2 (1974), 6 pp.
6. David Lee Hilliker, "A Study in the History of Analysis up to the Time of Leibniz and Newton in regard to Newton's Discovery of the Binomial Theorem," fourth part, "Contributions of Newton," *The Mathematics Student,* Vol. XLII, No. 1 (1974), 8 pp.
7. David Lee Hilliker, "On the infinite Multinomial Expansion," Second paper, *The Fibonacci Quarterly* (to appear).
8. David Lee Hilliker, "On the Multinomial Theorem," *The Fibonacci Quarterly* (to appear).
9. A. I. Markushevich, *Theory of Functions of a Complex Variable,* Volumes I, II, and III. Prentice-Hall, Englewood Cliffs, New Jersey, 1965. Translated from the Russian.

****