A MODEL FOR POPULATION GROWTH

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In 1969, Parberry [1] posed and solved an interesting problem in population growth analogous to the rabbit problem considered by Fibonacci. In this note we describe how one might treat a generalization of these problems. First, we state the problems of Fibonacci and Parberry and note what they have in common.

The situation considered by Fibonacci involves two types of rabbit which will be denoted B and F (for baby and female, respectively). Starting with one individual of type B, a sequence of generations of rabbits is formed as follows: Each individual of type B in the n^{th} generation matures to become an individual of type F in the $(n+1)^{st}$ generation. Also, each individual of type F in the n^{th} generation gives birth to an individual of type B in the $(n+1)^{st}$ generation, and survives to become an individual of type F in the $(n+1)^{st}$ generation. A family tree may be drawn which represents this process; see Figure 1.

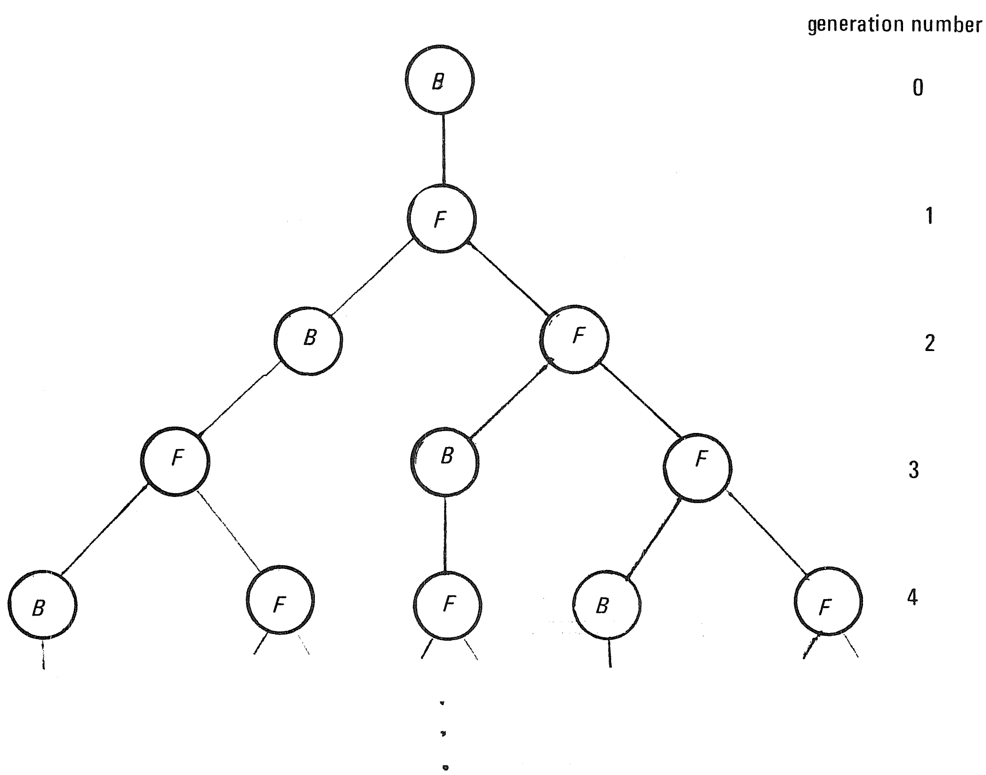

Figure 1. Family Tree of Fibonacci's Rabbits

Parberry considered populations of diatoms, one-celled algae whose reproductive capabilities can be classified according to size and maturity. Changes in classification along with reproduction are assumed to take place at regular intervals which will be called generations. Let m and n denote natural numbers, and let

$$S_1, \cdots, S_m, S_{m+1}, \cdots, S_{m+n}$$

denote a classification of the diatoms. Diatoms of type S_i for $i = 1, \cdots, m$ split to form two new diatoms, one of type S_i and the other of type S_{i+1}, but diatoms of type S_{m+i} for $i = 1, \cdots, n$ can only mature to become diatoms of type S_{m+i+1}, where a diatom of type S_{m+n+1} is defined to be of type S_1. For example, when $m = 2, n = 1$, the family tree of diatoms descending from one individual of type S_1 is shown in Figure 2.

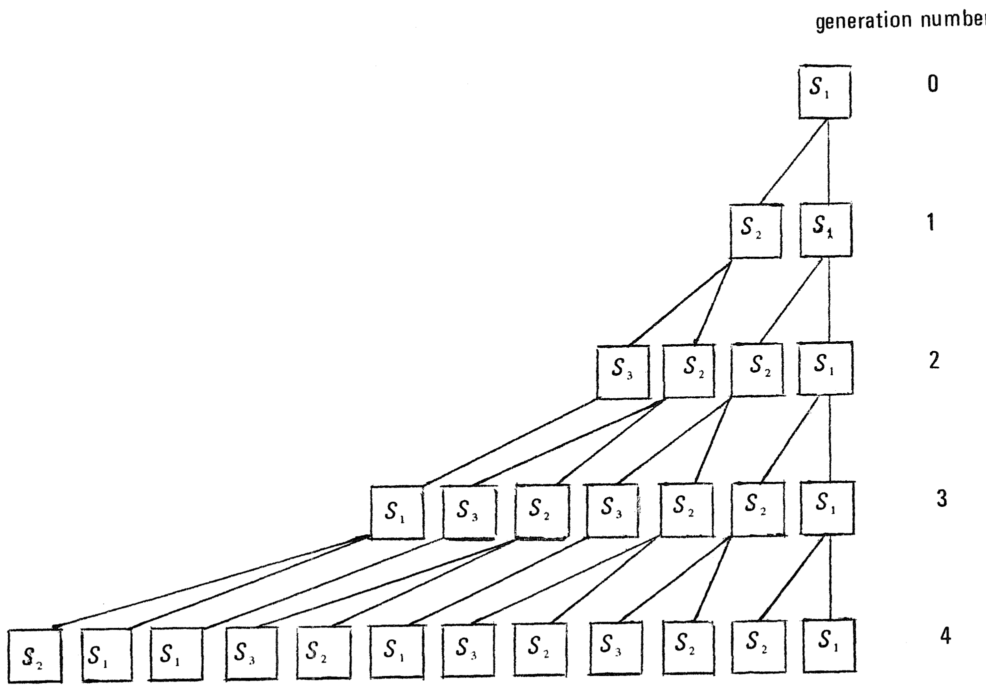

Figure 2. Family Tree of Diatoms

The problems of Fibonacci and Parberry have common features which are embodied in the following generalization. There is a finite set $T = \{1, \cdots, t\}$ of types of individuals, and each individual of type i in the n^{th} generation gives rise to f_{ij} individuals of type j in the $(n+1)^{st}$ generation $(1 \leqslant i, j \leqslant t)$ for $n = 0, 1, \cdots$. Also, there is an initial population containing f_i individuals of type i. Let $f_i(n)$ denote the number of individuals of type i in the n^{th} generation. (Thus, $f_i = f_2(0)$), and put

$$f(n) = f_1(n) + \cdots + f_t(n).$$

The sequences

$$(f_i(n) : n = 0, 1, \cdots)$$

are of interest: How are they related, how should they be calculated, what is their rate of growth, and so on. There is a very simple theory which explains these things.

Each of the $f_j(n-1)$ individuals of type j in the $(n-1)^{st}$ generation gives rise to f_{jk} individuals of type k in the n^{th} generation; hence, summing on j we have

(1) $$f_k(n) = f_1(n)f_{1k} + \cdots + f_t(n)f_{tk}.$$

This may be expressed in terms of matrices as

$$[f_1(n) \cdots f_t(n)] = [f_1(n-1) \cdots f_t(n-1)] \begin{bmatrix} f_{11} & f_{12} \cdots & f_{1t} \\ f_{21} & f_{22} \cdots & f_{2t} \\ \vdots & \vdots & \\ f_{t1} & f_{t2} \cdots & f_{tt} \end{bmatrix},$$

or, with an obvious notational convention, this may be succinctly expressed as

(2) $$\bar{f}(n) = \bar{f}(n-1)F.$$

Using (2), an easy induction argument gives

(3) $$\bar{f}(n) = \bar{f}(0)F^n.$$

Now (3) can be used to show that each of the sequences $(f_i(n) : n = 0, 1, \cdots)$ satisfies a certain difference equation, hence, the sequence $(f(n) : n = 0, 1, \cdots)$ also satisfies this difference equation. Recall the Cayley-Hamilton Theorem: Every square matrix M satisfies its characteristic equation. Thus, if we form the polynomial $c_F(x) = \det (xI - F)$, where I denotes the $t \times t$ identity matrix, then $c_F(F)$ is the all-zero matrix. Hence,

(4) $$F^n c_F(F) = 0$$

for $n = 0, 1, \cdots$. Let

$$c_F(x) = x^t - a_1x^{t-1} - \cdots - a_t,$$

and let $f_{ij}(n)$ denote the $(i,j)^{th}$ entry of F^n for $n = 0, 1, \cdots$. Then (4) implies

(5) $$f_{ij}(n+t) - a_1f_{ij}(n+t-1) - \cdots - a_tf_{ij}(n) = 0$$

for $n = 0, 1, \cdots$ and $1 \leqslant i, j \leqslant t$. Since each of the sequences

$$(f_{ij}(n) : n = 0, 1, \cdots)$$

satisfies the same difference equation given in (5), any linear combination of these sequences also satisfies this difference equation. In particular, this implies

(6) $$f_i(n+t) = a_1f_i(n+t-1) + \cdots + a_tf_i(n)$$

for $n = 0, 1, \cdots$; also, the sequence

$$(f(n) : n = 0, 1, \cdots)$$

satisfies this difference equation.

The matrix F may satisfy a polynomial equation with degree less than t; if so, this polynomial may be used in place of $c_F(x)$ to obtain a lower order difference equation. It is well known and easy to prove that there is a polynomial, unique up to a constant factor, having minimal degree such that F satisfies the corresponding polynomial equation. This polynomial, called the *minimal polynomial* of F, is a factor of $c_F(x)$.

Returning to Fibonacci's problem, the matrix involved is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} :$$

The minimal polynomial of this matrix is $x^2 - x - 1$: Hence, $f(n)$, the number of rabbits in the n^{th} generation satisfies

(7) $$f(n+2) - f(n+1) - f(n) = 0$$

for $n = 0, 1, \cdots$; also, we have $f(0) = f(1) = 1$, so this gives Fibonacci's sequence 1, 1, 2, 3, 5, 8, ⋯.

A more realistic model of a rabbit population would reflect the fecundity of the female depending on her age. For example, type 1 matures to become type 2; type 2 has a litter of 3 type 1's and matures to become type 3; type 3 has a litter of 4 type 1's and matures to become type 4; type 4 has a litter of 2 type 1's and dies. The matrix involved is

(8) $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

and the characteristic equation is

$$x^4 - 3x^2 - 4x - 2 = (x+1)(x^3 - x^2 - 2x - 2).$$

Suppose the initial population consists of one rabbit of type 1, then the family tree shown in Figure 3 results.

generation number

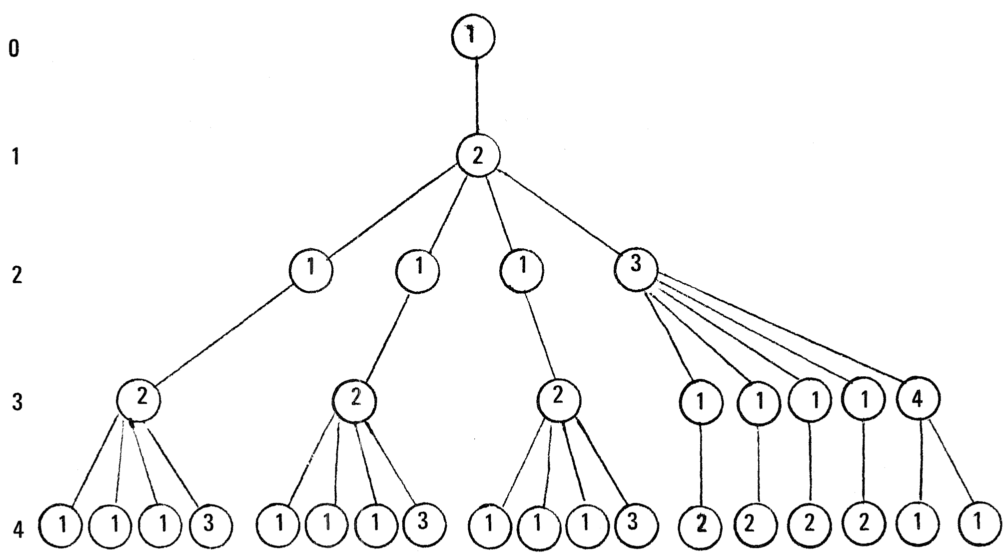

Figure 3

The number of rabbits in the n^{th} generation satisfies

(9) $$f(n+4) = 3f(n+2) + 4f(n+1) + 2f(n),$$

but it is easy to check that the initial conditions

$$f(0) = f(1) = 1, \quad f(2) = 4, \quad \text{and} \quad f(3) = 8$$

give rise to a sequence 1, 7, 4, 8, 18, ⋯ which satisfies the lower order difference equation

(10) $$f(n+3) = f(n+2) + 2f(n+1) + 2f(n).$$

This relation arose because

$$x^3 - x^2 - 2x - 2$$

is a factor of

$$x^4 - 3x^2 - 4x - 2.$$

Since

$$x^3 - x^2 - 2x - 2$$

has a real zero θ between 2.2 and 2.3, it follows that

$$f(n) > (2.2)^n$$

for all sufficiently large n.

REFERENCE

1. Edward A. Parberry, "A Recursion Relation for Populations of Diatoms," *The Fibonacci Quarterly*, Vol. 7, No. 4 (Dec. 1969), pp. 449–456.

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which tends to

$$\log \frac{1+\sqrt{5}}{2}$$

as $n \to \infty$ and this completes the proof.

In addition we want to mention another interesting property possessed by the sequences of the previous theorems. This property can be shown by applying a result of Vanden Eynden (see [2] p. 307): Let (C_n) be a sequence of real numbers such that the sequence (C_n/m) is u.d. mod 1 for all integers $m \geqslant 2$. Then the sequence $([C_n])$ of integral parts is u.d. in the ring of integers $\mathbb{Z}$.

Theorem 4. The sequences

$$([\log F_n^{1/k}]), \qquad ([\log H_n H_n^*]) \qquad \text{and} \qquad ([\log (H_n + H_n^*)])$$

are u.d. in $\mathbb{Z}$.

Proof. It is easily seen that for all non-zero integers m the expressions

$$\frac{1}{m}\log F_n^{1/k}, \quad \frac{1}{m}\log (H_n H_n^*) \quad \text{and} \quad \frac{1}{m}\log (H_n + H_n^*)$$

satisfy the condition in van der Corput's Theorem.

REFERENCES

1. William Webb, "Distribution of the First Digits of Fibonacci Numbers," *The Fibonacci Quarterly*, Vol. 13, No. 4 (Dec. 1975), pp. 334–336.
2. L. Kuipers and H. Niederreiter, "Uniform Distribution of Sequences," 1974.

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