THE SUMS OF CERTAIN SERIES CONTAINING HYPERBOLIC FUNCTIONS

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1. INTRODUCTION

In this paper we are concerned with the summation of a number of series. They are

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^{4p-1}\sinh r\pi}, \quad \sum_{r=1}^{\infty} \frac{\coth r\pi}{r^{4p-1}}, \quad \sum_{r=0}^{\infty} \frac{\tanh (2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}, \quad \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^{4p-3}\cosh (2r+1)\frac{\pi}{2}}$$

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1} r^{4p-3}}{\sinh r\pi}, \quad \sum_{r=0}^{\infty} \frac{(-1)^r (2r+1)^{4p-1}}{\cosh (2r+1)\frac{\pi}{2}}$$

and

$$\sum_{r=1}^{\infty} \frac{\left\{2^{4p}\coth r\frac{\pi}{2} - \coth 2r\pi\right\}}{r^{4p+1}},$$

where $p = 1, 2, 3, \cdots$.

Certain of the above series have been extensively discussed in the past. Results for particular values of p are given by Ramanujan in [4], while Phillips, Sandham and Watson in [3, 5, 6] have determined, by varying methods, sums for general p. The last series of the group, however, seems to have received less attention. It differs from the others in that it contains the inverse powers of $4p+1$. Further, it is closely related to the Riemann Zeta function $\zeta(4p+1)$. As this paper shows, the sums of the series, where they are not identically zero, satisfy recursive relations containing binomial coefficients.

Thus if we write

$$T_{4p-1} = \frac{(-1)^p (4p)!}{\pi^{4p-1} 2^{2p-2}} \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{4p-1}\sinh r\pi}$$

then

$$\sum_{p=1}^{n} \binom{4n+2}{4p} T_{4p-1} = 1 \qquad n = 1, 2, \cdots.$$

The recursive relations are themselves of interest and can be inverted. Their inversion, which leads to the sums of the various series, involves the Bernoulli and the lesser known Euler numbers.

All results are obtained by considering the Neumann problem for the rectangle. Although this problem is of an elementary nature and is in fact discussed in both contemporary and established literature on Laplace's equation, a complete solution to it does not seem to be available. Kantorovich and Krylov in [2] proposed a method of solution but the suggested method contains, as we shall show, an error of principle. Once this error is removed the method can be applied to solve the problem. Initially, therefore, we state and solve the Neumann problem for the rectangle and then subsequently in Section 3 make appropriate use of the solution to obtain the various results.

2. THE NEUMANN PROBLEM FOR THE RECTANGLE

This problem requires the determination of a function $\phi(x,y)$ satisfying

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{for} \quad 0 < x < a, \quad 0 < y < b \tag{2.1}$$

$$\phi_y(x,0) = f(x), \quad \phi_y(x,b) = g(x) \quad \text{for} \quad 0 \leqslant x \leqslant a \tag{2.2}$$

$$\phi_x(0,y) = F(y), \quad \phi_x(a,y) = G(y) \quad \text{for} \quad 0 \leqslant y \leqslant b, \tag{2.3}$$

where $f(x)$, $g(x)$, $F(y)$ and $G(y)$ are known functions and the subscripts x and y are used to denote partial differentiation.

It is necessary for a solution that

$$\int_c \frac{\partial \phi}{\partial n}\, ds = 0, \tag{2.4}$$

where c is the boundary of the rectangle, $\partial/\partial n$ denotes differentiation with respect to the outward normal to c and s refers to arc length. The condition (2.4) is equivalent to

$$\int_0^a (f - g)dx + \int_0^b (F - G)dy = 0. \tag{2.5}$$

We now briefly describe the method used by Kantorovich and Krylov in [2]. We put $\Phi = U + V$, where U and V are functions of x and y. We choose the function U so that it satisfies (2.1), (2.2) and $U_x(0,y) = U_x(a,y) = 0$ for $0 \leqslant y \leqslant b$, while V satisfies (2.1), (2.3) and $V_y(x,0) = V_y(x,b) = 0$ for $0 \leqslant x \leqslant a$.

Thus, the original Neumann problem is replaced by two other Neumann problems, one for U and the other for V. It is evident that if we can find U and V we shall fulfill the conditions imposed on ϕ by (2.1) to (2.3). By virtue of (2.4) the existence of U requires

$$\int_0^a (f - g)dx = 0.$$

Likewise, the existence of V requires

$$\int_0^b (F - G)dy = 0.$$

However, given functions f, g, F and G satisfying (2.5), it does not necessarily follow that the integrals

$$\int_0^a (f - g)dx \quad \text{and} \quad \int_0^b (F - G)dy$$

are each zero, and therefore the functions U and V may not exist. Yet the difficulty is readily overcome. We write

$$\phi = A(x^2 - y^2) + U + V,$$

where A is some constant to be found, while the functions U and V each satisfy (2.1) and the further conditions:

$$U_x(0,y) = U_x(a,y) = V_y(x,0) = V_y(x,b) = 0$$

$$U_y(x,0) = f(x), \quad U_y(x,b) = g(x) + 2Av \quad \text{for} \quad 0 \leqslant x \leqslant a$$

$$V_x(0,y) = F(y), \quad V_x(a,y) = G(y) - 2Aa \quad \text{for} \quad 0 \leqslant y \leqslant b.$$

Using (2.4) we require for the existence of U and V

$$\int_0^a \left\{ g(x) + 2Ab - f(x) \right\} dx = 0, \quad \text{i.e.,} \quad 2abA = \int_0^a (f - g)dx$$

and

$$\int_0^b \left\{ G(y) - 2Aa - F(y) \right\} dy = 0 \quad \text{or} \quad 2abA = \int_0^b (G-F)dy .$$

Equation (2.5) shows that these two expressions for A are consistent. Having found A, we can now follow the procedure given in [2] to determine U and V. In fact, it can be verified directly that to within an arbitrary constant Φ is given by

$$\phi = A(x^2 - y^2) + \tfrac{1}{2}f_0 y + \tfrac{1}{2}F_0 x + \sum_{r=1}^{\infty} \frac{a\left\{ g_r \cosh \frac{r\pi y}{a} - f_r \cosh \frac{r\pi}{a}(b-y) \right\}}{r\pi \sinh \frac{r\pi b}{a}} \cos \frac{r\pi x}{a} \tag{2.6}$$

$$+ \sum_{r=1}^{\infty} \frac{b\left\{ G_r \cosh \frac{r\pi x}{b} - F_r \cosh \frac{r\pi}{b}(a-x) \right\}}{r\pi \sinh \frac{r\pi a}{b}} \cos \frac{r\pi y}{b} ,$$

where f_r, g_r $(r = 0, 1, 2, \cdots)$ are the Fourier cosine coefficients for $f(x)$ and $g(x)$, respectively, over the range $0 \leqslant x \leqslant a$ and F_r, G_r $(r = 0, 1, 2, \cdots)$ are the Fourier cosine coefficients of $F(y)$ and $G(y)$ over $0 \leqslant y \leqslant b$.

3. APPLICATION OF THE SOLUTION TO THE NEUMANN PROBLEM

We put $a = b = \pi$ and define functions $\phi(x,y,4n)$, where $n = 1, 2, 3, \cdots$, by

$$2\phi(x,y,4n) = (x+iy)^{4n} + (x-iy)^{4n} . \tag{3.1}$$

It is readily verified that these functions satisfy (2.1). Further, using (2.2) and (2.3), we deduce for them that $f(x)$ and $F(y)$ are both identically zero. In addition

$$g(x) = 2n\left\{ (\pi+ix)^{4n-1} + (\pi-ix)^{4n-1} \right\} \quad \text{and} \quad G(y) = 2n\left\{ (\pi+iy)^{4n-1} + (\pi-iy)^{4n-1} \right\} .$$

Thus, the Fourier coefficients f_r and F_r are all zero, while $g_r = G_r = I_r(n)$ $(r = 1, 2, \cdots)$, where

$$I_r(n) = Re\, \frac{4n}{\pi} \int_0^{\pi} [(\pi+ix)^{4n-1} + (\pi-ix)^{4n-1}] e^{irx} dx \tag{3.2}$$

using the result

$$2abA = \int_0^a (f-g)dx$$

we find that the constant A vanishes and hence with the help of (2.6) we can write

$$\phi(x,y,4n) = c_{4n} + \sum_{r=1}^{\infty} I_r(n) \frac{\left\{ \cosh ry \cos rx + \cosh rx \cos ry \right\}}{r \sinh r\pi} \tag{3.3}$$

the c_{4n} $(n = 1, 2, \cdots)$ being constants which have yet to be determined. Successive integration by parts of (3.2) leads to the result

$$I_r(n) = \frac{(-1)^{n+r}}{r^2} \pi^{4n-3} 2^{2n} (4n)(4n-1) + \frac{4n}{r^4} (4n-1)(4n-2)(4n-3) I_r(n-1) . \tag{3.4}$$

In particular

$$I_r(1) = (-1)^{r+1} \frac{48\pi}{r^2}$$

so that putting $n = 1$ in (3.1) and (3.3) we find

$$x^4 - 6x^2y^2 + y^4 = c_4 + 48\pi \sum_{r=1}^{\infty} (-1)^{r+1} \frac{\left\{ \cosh ry \cos rx + \cosh rx \cos ry \right\}}{r^3 \sinh r\pi} . \tag{3.5}$$

Repeated application of (3.4) yields

$$I_r(n) = (-1)^r \left\{ \frac{a_2(n)}{r^2} + \frac{a_6(n)}{r^6} + \frac{a_{10}(n)}{r^{10}} + \frac{a_{4n-2}(n)}{r^{4n-2}} \right\}, \tag{3.6}$$

where, for example,

$$a_2(n) = (-1)^n \pi^{4n-3} 2^{2n} 4n(4n-1) \tag{3.7}$$

and more generally

$$a_{4p-2}(n) = (-1)^{n-p+1} \pi^{4n-4p+1} 2^{2n-2p+2} (4p-2)! \binom{4n}{4p-2}, \quad p = 1, 2, \cdots, n. \tag{3.8}$$

Using this last result, it follows

$$a_{4p+2}(n+1) = (4n+4)(4n+3)(4n+2)(4n+1)a_{4p-2}(n)$$

and hence from (3.6) that

$$(4n+4)(4n+3)(4n+2)(4n+1)\frac{I_r(n)}{r^4} = I_r(n+1) + \frac{(-1)^{r+1}}{r^2} a_2(n+1). \tag{3.9}$$

We now proceed to find the constants c_{4n} occurring in (3.3). We integrate Eq. (3.3) twice with respect to x and twice with respect to y. These integrations will introduce arbitrary functions of x and y. We have, therefore,

$$\frac{-\phi(x,y,4n+4)}{(4n+1)(4n+2)(4n+3)(4n+4)} + xP_n(y) + Q_n(y) + yp_n(x) + q_n(x)$$

$$= c_{4n}\frac{x^2y^2}{4} - \sum_{r=1}^{\infty} I_r(n) \frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r^5 \sinh r\pi},$$

where $p_n(x)$, $q_n(x)$, $P_n(y)$ and $Q_n(y)$ are arbitrary functions which may depend on n. Noting the result contained in (3.9) we can write this equation in the alternative form

$$\phi(x,y,4n+4) = \sum_{r=1}^{\infty} \left\{ I_r(n+1) + \frac{(-1)^{r+1}}{r^2} a_2(n+1) \right\} \frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r \sinh r\pi}$$

$$+ (4n+1)(4n+2)(4n+3)(4n+4) \left\{ xP_n(y) + Q_n(y) + yp_n(x) + q_n(x) - c_{4n}\frac{x^2y^2}{4} \right\}.$$

This reduces with the help of (3.3) and (3.5) to

$$0 = -c_{4n+4} + \frac{a_2(n+1)}{48\pi}(x^4 - 6x^3y^2 + y^4 - c_4)$$

$$+ (4n+1)(4n+2)(4n+3)(4n+4) \left\{ xP_n(y) + Q_n(y) + yp_n(x) + q_n(x) - c_{4n}\frac{x^2y^2}{4} \right\}.$$

This is an identity. Hence equating to zero the coefficient of x^2y^2 we deduce with the aid of (3.7)

$$c_{4n} = \frac{(-1)^n \pi^{4n} 2^{2n}}{(2n+1)(4n+1)}. \tag{3.10}$$

Thus we have

$$(x+iy)^{4n} + (x-iy)^{4n} = 2c_{4n} + 2\sum_{r=1}^{\infty} I_r(n) \frac{\{\cosh ry \cos rx + \cosh rx \cos ry\}}{r \sinh r\pi}, \tag{3.11}$$

where c_{4n} is given by (3.10) and $I_r(n)$ by results (3.6) and (3.8). Putting $x = y = 0$ in (3.11) and simplifying we obtain

$$0 = \frac{1}{(4n+1)(4n+2)} + \sum_{r=1}^{\infty} \frac{(-1)^r}{r \sinh r\pi} \left\{ \sum_{p=1}^{n} \frac{(-1)^{p-1}(4p-2)! \binom{4n}{4p-2}}{\pi^{4p-1} 2^{2p-2} r^{4p-2}} \right\} \quad n = 1, 2, \cdots.$$

Thus if we write

$$(3.12)\qquad T_{4p-1} = \frac{(-1)^p(4p)!}{\pi^{4p-1}2^{2p-2}} \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{4p-1}\sinh r\pi}, \quad p = 1, 2, \cdots$$

then it follows T_{4p-1} satisfies the recursive relation

$$(3.13)\qquad 1 = \sum_{p=1}^{n} \binom{4n+2}{4p} T_{4p-1} \quad n = 1, 2, \cdots .$$

This is the first of our results. We now show how this recursive relation can be inverted to give T_{4p-1} in terms of the Bernoulli numbers. To do this, we observe that (3.13) can be put in the alternative form

$$\frac{1}{(4n+2)!} = \sum_{p=1}^{n} \frac{T_{4p-1}}{(4p)!(4n+2-4p)!} .$$

Multiplying both sides of this equation by x^{4n+2} and summing from $n = 1$ to ∞ yields

$$\sum_{n=1}^{\infty} \frac{x^{4n+2}}{(4n+2)!} = \sum_{n=1}^{\infty}\sum_{p=1}^{n} \frac{T_{4p-1}x^{4n+2}}{(4p)!(4n+2-4p)!} = \left\{\sum_{p=1}^{\infty} \frac{T_{4p-1}x^{4p}}{(4p)!}\right\}\left\{\sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}\right\} .$$

After some manipulation we obtain

$$(3.14)\qquad \sum_{p=1}^{\infty} T_{4p-1}\frac{x^{4p}}{(4p)!} = 1 - \frac{x^2}{\cosh x - \cos x} = 1 + \frac{x^2}{2}\operatorname{cosech} \alpha x \operatorname{cosech} i\alpha x,$$

where $2\alpha = 1 + i$.

Using the expansion of cosech x given in [1] Eq. (3.15) leads after some simplification to

$$T_{4p-1} = \frac{(-1)^{p+1}}{2^{2p-2}} \sum_{q=0}^{2p} (-1)^2(2^{2q-1}-1)(2^{4p-2q-1}-1)\binom{4p}{2q} B_q B_{2p-q} .$$

It should be noted that B_0 is taken as -1 while the Bernoulli numbers are defined here by

$$\frac{x}{e^x-1} = 1 - \frac{x}{2} + \sum_{p=1}^{\infty} (-1)^{p+1}B_p \frac{x^{4p}}{(2p)!} .$$

With the help of (3.12) we deduce

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^{4p-1}\sinh r\pi} = \pi^{4p-1} \sum_{q=0}^{2p} \frac{(-1)^q(2^{2q-1}-1)(2^{4p-2q-1}-1)}{(2q)!\,(4p-2q)!} B_q B_{2p-q} .$$

In a similar manner if we put $x = y = \pi$ in (3.11) and define s_{4p-1} by

$$s_{4p-1} = (-1)^{p-1}\pi^{1-4p}2^{-2p+2}(4p)! \sum_{r=1}^{\infty} \frac{\coth r\pi}{r^{4p-1}}$$

then

$$(3.16)\qquad \sum_{p=1}^{n} \binom{4n+2}{4p} s_{4p-1} = 2n(4n+3)$$

s_{4p-1} can also be expressed in terms of the Bernoulli numbers. By writing (3.16) in the form

$$\sum_{p=1}^{n} \frac{s_{4p-1}}{(4n-4p+2)!(4p)!} = \frac{1}{2}\left\{\frac{1}{(4n)!} - \frac{2}{(4n+2)!}\right\}$$

and following a procedure similar to that for T_{4p-1} we find

$$\sum_{p=1}^{\infty} s_{4p-1} \frac{x^{4p}}{(4p)!} = -\frac{x^2}{2} \coth \alpha x \coth i\alpha x - 1.$$

Since (see [1]),

$$x \coth x = \sum_{p=0}^{\infty} (-1)^{p+1} B_p 2^{2p} \frac{x^{2p}}{(2p)!}$$

we have

$$s_{4p-1} = 2^{2p} \sum_{q=0}^{2p} (-1)^{p+q} \binom{4p}{2q} B_q B_{2p-q}$$

giving

(3.17)
$$\sum_{r=1}^{\infty} \frac{\coth r\pi}{r^{4p-1}} = 2^{4p-2}\pi^{4p-1} \sum_{q=0}^{2p} \frac{(-1)^{q+1} B_q B_{2p-q}}{(2q)!(4p-2q)!}.$$

We next put $x = 0$, $y = \pi$ in (3.11) and subtract from twice the result the expressions obtained by putting $x = y = 0$ and $x = y = \pi$. This leads to

(3.18)
$$\pi^{4n}[1 + (-1)^{n+1} 2^{2n-1}] = \sum_{r=0}^{\infty} 2 I_{2r+1}(n) \frac{\tanh (2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}.$$

Writing

(3.19)
$$Q_{4p-1} = (-1)^{p-1} \pi^{-4p+1} 2^{-2p+4} (4p)! \sum_{r=0}^{\infty} \frac{\tanh (2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}$$

(3.18) gives with the aid of (3.6) and (3.19)

(3.20)
$$\sum_{p=1}^{n} \binom{4n+2}{4p} Q_{4p-1} = (4n+1)(4n+2)\left\{1 + (-1)^{n+1} 2^{1-2n}\right\}.$$

This is the third of the recursive relations and may be compared directly in form with (3.13) and (3.16). Q_{4p-1} can also be expressed in terms of the Bernoulli numbers.

From (3.20) we deduce

$$x^2 \sum_{n=1}^{\infty} \left\{1 + (-1)^{n+1} 2^{1-2n}\right\} \frac{x^{4n}}{(4n)!} = \left\{\sum_{p=1}^{\infty} Q_{4p-1} \frac{x^{4p}}{(4p)!}\right\}\left\{\sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}\right\}$$

or, after some manipulation,

$$\sum_{p=1}^{\infty} Q_{4p-1} \frac{x^{4p}}{(4p)!} = x^2 \frac{\{\cosh x + \cos x - 2\cosh \alpha x - 2\cos \alpha x + 2\}}{\cosh x - \cos x},$$

where as before $2\alpha = 1 + i$.

The right-hand side of (3.21) can be expressed as

$$\frac{x^2}{2}\left\{\coth\frac{ax}{2}\coth\frac{iax}{2}-2\coth ax\coth iax-2\operatorname{cosec} ax\operatorname{cosech} iax-\tanh\frac{ax}{2}\tanh\frac{iax}{2}\right\}.$$

Recalling the expansions for $\coth x$, $\operatorname{cosech} x$ already used and noting that in [1] for $\tanh x$ we obtain after some manipulation

$$Q_{4p-1}=\frac{(-1)^p(4p)!}{2^{2p-3}}\sum_{q=0}^{2p}(-1)^q\frac{(2^{4p-2q}-1)(2^{2q}-1)}{(2q)!(4p-2q)!}B_qB_{2p-q}$$

and hence by (3.19)

$$\sum_{r=1}^{\infty}\frac{\tanh(2r+1)\frac{\pi}{2}}{(2r+1)^{4p-1}}=\frac{\pi^{4p-1}}{2}\sum_{q=1}^{2p-1}(-1)^{q+1}\frac{(2^{4p-2q}-1)(2^{2q}-1)}{(2q)!(4p-2q)!}B_qB_{2p-q}. \tag{3.22}$$

The expression in (3.11) can be differentiated as many times as we wish with respect to x and y at points within the rectangle.

Differentiating once with respect to x and once with respect to y gives

$$(2n)(4n-1)i[(x+iy)^{4n-2}-(x-iy)^{4n-2}]=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}r}{\sinh r\pi}[\sinh ry\sin rx+\sinh rx\sin ry] \tag{3.23}$$

$$\times\left\{\sum_{p=1}^{n}(-1)^{-p+n+1}\pi^{4n-4p+1}\frac{2^{2n-2p+2}(4p-2)!}{r^{4p-2}}\binom{4n}{4p-2}\right\}.$$

Putting $x=y=\pi/2$ in (3.23) and defining R_{4p-3} by

$$R_{4p-3}=\frac{(-1)^{p-1}}{\pi^{4p-3}2^{2p-1}}(4p-2)!\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^{4p-3}\cosh(2r+1)\frac{\pi}{2}} \tag{3.24}$$

yields

$$\frac{(4n)(4n-1)}{2^{4n}}=\sum_{p=1}^{n}\binom{4n}{4p-2}R_{4p-3}. \tag{3.25}$$

The quantities R_{4p-3} can be expressed in terms of the Euler numbers (see [1]).

Following a procedure similar to earlier ones, we can deduce from (3.25) that

$$\sum_{p=0}^{\infty}R_{4p+1}\frac{x^{4p}}{(4p+2)!}=\frac{1}{2^4}\sec\frac{ax}{2}\sec\frac{iax}{2}.$$

Since

$$\sec x=\sum_{q=0}^{\infty}E_q\frac{x^{2q}}{(2q)!},$$

where $E_1, E_2, \cdots$, are the Euler numbers and E_0 is taken as unity, we obtain

$$R_{4p+1}=(4p+2)!\frac{(-1)^p}{2^{6p+4}}\sum_{q=0}^{2p}(-1)^q\frac{E_qE_{2p-q}}{(2q)!(4p-2q)!} \tag{3.26}$$

and hence

$$(3.27)\qquad \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^{4p+1}\cosh (2r+1)\frac{\pi}{2}} = \pi^{4p+1}2^{-4p-3}\sum_{q=0}^{2p}\frac{(-1)^q E_q E_{2p-q}}{(2q)!(4p-2q)!}, \quad p = 0, 1, 2, \cdots.$$

Putting $n = 1$ in (3.23) yields

$$(3.28)\qquad xy = 2\pi \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r\sinh r\pi}\left\{\sinh ry \sin rx + \sinh rx \sin ry\right\}.$$

Hence differentiating once with respect to x and then y we have, on putting $x = y = \pi/2$

$$1 = 4\pi \sum_{r=1}^{\infty} \frac{r(-1)^{r+1}}{\sinh r\pi}.$$

If we differentiate (3.28) $(2p + 1)$ times with respect to x and $(2p + 1)$ times with respect to y then for $x = y = \pi/2$ we find

$$\sum_{r=1}^{\infty} \frac{r^{4p+1}(-1)^r}{\sinh r\pi} = 0, \quad p = 1, 2, \cdots.$$

Likewise differentiating (3.28) $(2p)$ times with respect to x and $2p$ times with respect to y leads to

$$\sum_{r=0}^{\infty} \frac{(2r+1)^{4p-1}(-1)^r}{\cosh (2r+1)\frac{\pi}{2}} = 0, \quad p = 1, 2, \cdots.$$

We now proceed to find the sum of the last of the series referred to in the Introduction. Using the results of Section 2, it can be shown for $n = 1, 2, \cdots$

$$(3.29)\qquad \frac{1}{2}\left\{(x+iy)^{4n+2} + (x-iy)^{4n+2}\right\} = (-1)^n\pi^{4n}2^{2n}(x^2-y^2) + \sum_{r=1}^{\infty}(-1)^r\,\frac{\left\{\cosh rx\cos ry - \cosh ry \cos rx\right\}}{r\sinh r\pi}$$

$$\times \left\{\sum_{p=1}^{n} (-1)^{n+1-p}\,\frac{\pi^{4n-4p+1}}{r^{4p}}\,2^{2n-2p+2}(4p)!\binom{4n+2}{4p}\right\}.$$

The constant appearing in the Neumann solution is determined here to be zero by observing that each side of (3.29) vanishes when $x = y = 0$.

Putting $x = \pi$, $y = 0$ in (3.29) and defining M_{4p+1} by

$$(3.30)\qquad M_{4p+1} = (-1)^{p+1}\pi^{-4p-1}2^{-2p+2}(4p)!\sum_{r=1}^{\infty}\frac{1+(-1)^{r+1}\cosh r\pi}{r^{4p+1}\sinh r\pi}, \quad p = 1, 2, \cdots$$

leads to

$$(3.31)\qquad \sum_{p=1}^{n} M_{4p+1}\binom{4n+2}{4p} = 1 + (-1)^{n+1}2^{-2n}.$$

From the recurrence relation (3.31) we deduce

$$\sum_{p=1}^{\infty} M_{4p+1}\frac{x^{4p}}{(4p)!} = 1 + \frac{i}{2}\cot\frac{\alpha x}{2}\tan\frac{i\alpha x}{2} - \frac{i}{2}\tan\frac{\alpha x}{2}\cot\frac{i\alpha x}{2}$$

and hence

$$M_{4p+1} = (-1)^p \frac{(4p)!}{2^{2p-2}} \sum_{s=0}^{2p} (-1)^s \frac{(2^{4p-2s+2}-1)}{(2s)!(4p-2s+2)!} B_s B_{2p+1-s} .$$

Since

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\cosh r\pi + 1}{r^{4p+1}\sinh r\pi} = \sum_{r=1}^{\infty} \left\{ \frac{\coth \frac{r\pi}{2} - 2^{-4p}\coth 2r\pi}{r^{4p+1}} \right\}$$

we can, noting (3.30), obtain the required sum. It also follows for $p \geqslant 3$ we can obtain a good approximation to

$$\sum_{r=1}^{\infty} \frac{\coth \frac{r\pi}{2}}{r^{4p+1}}$$

in terms of the Bernoulli numbers.

ACKNOWLEDGEMENTS

I am indebted to Professor L. Carlitz for pointing out to me how the result (3.17) can be deduced from (3.16). This result, I am told by Professor H. Halberstam, is contained, in a somewhat disguised form, in one of Ramanujan's notebooks. I also thank Bruce Berndt for drawing my attention to the papers by Phillips and Sandham.

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