ON THE INFINITE MULTINOMIAL EXPANSION, II

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In a previous note (Hilliker [7]) we derived, by an iterative argument, the following version of the Multinomial Expansion: If the inequalities

$$(1) \qquad |a_j| < |a_1 + a_2 + \cdots + a_{j-1}|,$$

for $j = 2, 3, \cdots, r$ all hold, then

$$(2) \qquad \left(\sum_{i=1}^{r} a_i\right)^n = \sum \frac{n(n-1)\cdots(n-n_1-n_2-\cdots-n_{r-1}+1)}{n_1!\,n_2!\cdots n_{r-1}!} a_r^{n_1} a_{r-1}^{n_2} \cdots a_2^{n_{r-1}} a_1^{n-n_1-n_2-\cdots-n_{r-1}},$$

where the summation is an iterated summation taken under all $n_i \geqslant 0$, where i first takes on the value $r-1$, then $r-2$, and so on until the last value, 1, is taken on. Here $n, a_1, a_2, \cdots, a_r$ are complex numbers with n not equal to a non-negative integer. On the other hand, one can assume a single inequality

$$(3) \qquad |a_2 + a_3 + \cdots + a_r| < |a_1|$$

and avoid the more complicated iterative argument by direct employment of the Multinomial Theorem for non-negative integral exponents. The result is that the same formal expansion (2) holds, but this time the summation is taken under all $n_i \geqslant 0$ with $n_1 + n_2 + \cdots + n_{r-1} = j$ for $j = 0, 1, 2, \cdots$. See, for example, Chrystal [2], where a similar version is established. In this note we shall view these two forms from the perspective of a single Multinomial Expansion valid under a certain divisibility condition on r.

Let p *be an integer with* $1 \leq p \leq r-1$, *and assume that the congruence*

$$(4) \qquad r \equiv 1 \pmod{p}$$

holds. If the inequalities

$$(5) \qquad |a_{r-(i+1)p+1} + a_{r-(i+1)p+2} + \cdots + a_{r-ip}| < |a_1 + a_2 + \cdots + a_{r-(i+1)p}|,$$

for $i = 0, 1, 2, \cdots q$, *all hold, where the non-negative integer* q *is given by* $r = 1 + (q+1)p$, *then the formal expansion (2) holds. Here the summation is taken under all* $n_i \geqslant 0$, $1 \leqslant i \leqslant r-1$, *with*

$$(6) \qquad n_{jp+1} + n_{jp+2} + \cdots + n_{jp+p} = t_j,$$

where $t_j = 0, 1, 2, \cdots$, *and where* j *first takes on the value* q *then* $q-1$, *and so on until the last value, 0, is taken on.*

Our argument rests upon Abel's proof of about 1825 of the Binomial Theorem:

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

for n and z complex and with $|z| < 1$. See Abel [1]. See also Markushevich [9], I, for this Maclaurin expansion. Here, as usual, we define z^n as being that branch of the function $f(z) = e^{n \log z}$ defined over the complex z-plane with the non-positive real axis excluded, and with $f(1) = 1$. That is, the logarithmic function is given by $\log z = \log |z| + i \arg z$ with $|\arg z| < \pi$. The quantities $a_1 + a_2 + \cdots + a_r$ and a_1 are not 0 by the inequalities (5) with $i = 0$ and $i = q$, respectively. We will need to assume that they are not negative real numbers. Likewise, in the course of the proof we will need to assume that the quantities $a_1 + a_2 + \cdots + a_{r-(i+1)p}$, for $0 \leqslant i \leqslant q-1$, are not negative real numbers. If n is a (negative) integer, these restrictions which guarantee single-valuedness, may, of course, be ignored.

As a first example, let $p = 1$. Then (4) automatically holds and $q = r - 2$. The inequalities (5) become identical with those of (1), and the summation conditions (6) become $n_{j+1} = t_j$ for $j = r-2, r-1, \cdots, 0$. Thus the first mentioned form is covered.

As a second example, let $p = r - 1$. Then (4) holds, and $q = 0$. The inequalities (5) reduce to the single inequality (3). The summation conditions (6) reduce to the single condition $n_1 + n_2 + \cdots + n_{r-1} = t_0$. Consequently, the second mentioned form is also covered.

We begin by writing

$$(7)\qquad (a_1 + a_2 + \cdots + a_r)^n = [(a_1 + a_2 + \cdots + a_{r-p}) + (a_{r-p+1} + a_{r-p+2} + \cdots + a_r)]^n$$
$$= \sum_{t_0=0}^{\infty} \binom{n}{t_0} \left(\sum_{k=r-p+1}^{r} a_k \right)^{t_0} \left(\sum_{\ell=1}^{r-p} a_\ell \right)^{n-t_0} .$$

Here we have used the inequality (5) for the case $i = 0$.

Since $n - t_0 \neq 0$, we may apply Formula (7) to the summation under ℓ on the right side of (7). We may repeat this iterative process. After m iterations of (7), $m \geqslant 0$ and not too large, one obtains, by using (5) for $i = 0, 1, \cdots, m$,

$$(8)\qquad (a_1 + a_2 + \cdots + a_r)^n = \sum_{t_0, t_1, \cdots, t_m = 0}^{\infty} \prod_{j=0}^{m} \binom{n - t_0 - \cdots - t_{j-1}}{t_j} \left(\sum_{k=r-(j+1)p+1}^{r-jp} a_k \right)^{t_j}$$
$$\times \left(\sum_{\ell=1}^{r-(m+1)p} a_\ell \right)^{n-t_0-t_1-\cdots-t_m} .$$

First we apply the Multinomial Theorem for non-negative integral exponents to the summation under k on the right side of (8). Since this summation contains p terms, we can write

$$(9)\qquad \left(\sum_{k=r-(j+1)p+1}^{r-jp} a_k \right)^{t_j} = \sum \frac{t_j!}{n_{jp+1}!\, n_{jp+2}! \cdots n_{jp+p}!} a_{r-jp}^{n_{jp+1}} a_{r-jp-1}^{n_{jp+2}} \cdots a_{r-jp-p+1}^{n_{jp+p}} ,$$

where the summation is taken under all non-negative values of the p integers $n_{jp+1}, n_{jp+2}, \cdots, n_{jp+p}$ subject to the restruction (6).

Secondly we observe that

$$(10)\qquad \prod_{j=0}^{m} \binom{n - t_0 - t_1 - \cdots - t_{j-1}}{t_j} = \frac{n(n-1)\cdots(n - n_1 - n_2 - \cdots - n_{mp+p} + 1)}{t_0!\, t_1! \cdots t_m!}$$

since, by (6), $t_0 + t_1 + \cdots + t_m = n_1 + n_2 + \cdots + n_{mp+p}$.

Finally we note that from (4) we can choose m in such a way that $r - (m+1)p = 1$, so that the summation under ℓ on the right side of (8) reduces to a single term.

Thus it follows from (8), (9) and (10) that

$$(a_1 + a_2 + \cdots + a_r)^n = \sum_{t_0, t_1, \cdots, t_m = 0}^{\infty} \frac{n(n-1)\cdots(n - n_1 - n_2 - \cdots - n_{r-1} + 1)}{n_1!\, n_2! \cdots n_{r-1}!}$$
$$\times a_r^{n_1} a_{r-1}^{n_2} \cdots a_2^{n_{r-1}} a_1^{n-n_1-n_2-\cdots-n_{r-1}} ,$$

where the summation is first taken under t_m, then under t_{m-1}, and so on until the last summation is taken under t_0.

Our expository sequence of papers on the Binomial Theorem, the Multinomial Theorem, and various Multinomial Expansions (Hilliker [3], [4], [5], [6], [7] and the present paper) will continue (Hilliker [8]).

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Let q^b denote one of the $p_i^{a_i}$ and P denote $q^{b-2}(q-1)^2$. Now,

$$q^{b-2}(q-1)^2 = q^{b-1}(q-2+1/q). \tag{3}$$

From (3), it can be seen that $P > 1$, for all q, and that $P > 8$, for all $q \geqslant 11$. Furthermore, for $q < 11$, the following table can be obtained, by checking the right side of (3) for the case $b = 1$, and the left side of (3) for the case $b \geqslant 2$.

Prime q	3	3	5	5	7	7
Exponent b	2	3	1	2	1	2
P greater than or equal to	4	8	2	8	4	8

Hence, (2) holds for $p-1$ possibly equal to $2\cdot 3$, $2\cdot 3^2$, $2\cdot 5$, $2\cdot 7$, $2\cdot 3\cdot 5$, $2\cdot 3\cdot 7$ ($a = 1$); $4\cdot 3$, $4\cdot 5$, $4\cdot 3\cdot 5$ ($a = 2$); or $8\cdot 3$ ($a = 3$); and (2) fails to hold for all other choices. These combinations lead to the primes 7, 11, 13, 19, 31, 43, 61.

Theorem 3. If p is a prime greater than 5, then the primitive roots are not consecutive.

Proof. For the primes excluded in the Lemma, the primitive roots are: for 7 – 3, 5; for 11 – 2, 6, 7, 8; for 13 – 2, 6, 7, 11; for 19 – 2, 3, 10, 13, 14, 15; for 31 – 3, 11, 12, 13, 17, 21, 22, 24; for 43 – 3, 5, 12, 18, 19, 20, 26, 28, 29, 30, 33, 34; for 61 – 2, 6, 7, 10, 17, 18, 26, 30, 31, 35, 43, 44, 51, 54, 55, 59. None of these primes have consecutive primitive roots.

Now, let p denote a prime for which the Lemma applies and suppose that k is a positive integer for which $k^2 \leqslant p-1$. Then,

$$k^2 - (k-1)^2 = 2\cdot k - 1 < 2\cdot k \leqslant 2\sqrt{p-1} \leqslant \phi(p-1).$$

Therefore, consecutive squares appear within a span less than $\phi(p-1)$. Since squares are quadratic residues, and therefore not primitive roots, no string of consecutive primitive roots can be of length $\phi(p-1)$. Consequently, the primitive roots are not consecutive.

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