SOME SUMS CONTAINING THE GREATEST INTEGER FUNCTION

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1. Let $[x]$ denote the greatest integer less than or equal to the real number x. It is well known (see for example [4, p. 97] that

$$\sum_{r=1}^{k-1} \left[\frac{hr}{k}\right] = \tfrac{1}{2}(h-1)(k-1), \tag{1.1}$$

where $(h,k) = 1$. Indeed

$$\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right] = \sum_{r=1}^{k-1}\left[\frac{h(k-r)}{k}\right] = \sum_{r=1}^{k-1}\left[h-\frac{hr}{k}\right] = \sum_{r=1}^{k-1}\left(h-1-\left[\frac{hr}{k}\right]\right)$$

$$= (h-1)(k-1) - \sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]$$

and (1.1) follows immediately.

For a later purpose we shall require the following extension of (1.1):

$$\sum_{r=0}^{k-1}\left[x+\frac{hr}{k}\right] = [kx] + \tfrac{1}{2}(h-1)(k-1). \tag{1.2}$$

For $h = 1$, (1.2) reduces to the familiar result [4, p. 97]

$$\sum_{r=0}^{k-1}\left[x+\frac{r}{k}\right] = [kx]\ . \tag{1.3}$$

To prove (1.2), put

$$\phi(x) = x - [x]\,, \tag{1.4}$$

the fractional part of x. Then clearly

$$\phi(x+1) = \phi(x) \tag{1.5}$$

and, by (1.3),

$$\sum_{r=0}^{k-1}\phi\left(x+\frac{r}{k}\right) = kx + \tfrac{1}{2}(k-1) - [kx] = \phi(kx) + \tfrac{1}{2}(k-1). \tag{1.6}$$

It follows, using (1.5), that

$$\sum_{r=0}^{k-1}\left[x+\frac{hr}{k}\right] = \sum_{r=0}^{k-1}\left(x+\frac{hr}{k}-\phi\left(x+\frac{hr}{k}\right)\right) = kx + \tfrac{1}{2}h(k-1) - \phi(kx) - \tfrac{1}{2}(k-1)$$

$$= [kx] + \tfrac{1}{2}(h-1)(k-1).$$

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The writer has recently proved [2] the following result:

$$6k\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^2+6h\sum_{s=1}^{h-1}\left[\frac{ks}{h}\right]^2=(h-1)(2h-1)(k-1)(2k-1),\tag{1.7}$$

where $(h,k)=1$. This formula can be proved rapidly in the following way.

Put

$$S_2(h,k)=\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^2=\sum_{r=0}^{k-1}\left[\frac{hr}{k}\right]^2.$$

We have

$$\sum_{r=0}^{k-1}\phi^2\left(\frac{hr}{k}\right)=\sum_{r=0}^{k-1}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]\right)^2=\frac{1}{6k}h^2(k-1)(2k-1)-\frac{2h}{k}\sum_{r=1}^{k-1}r\left[\frac{hr}{k}\right]+S_2(h,k).$$

Since, by (1.5),

$$\sum_{r=0}^{k-1}\phi^2\left(\frac{hr}{k}\right)=\sum_{r=0}^{k-1}\phi^2\left(\frac{r}{k}\right)=\sum_{r=0}^{k-1}\left(\frac{r}{k}\right)^2=\frac{1}{6k}(k-1)(2k-1),$$

it follows that

$$6kS_2(h,k)=12h\sum_{r=1}^{k-1}r\left[\frac{hr}{k}\right]-(h^2-1)(k-1)(2k-1).$$

It is known [3, p. 9] that

$$12h\sum_{r=1}^{k-1}r\left[\frac{hr}{k}\right]+12k\sum_{s=1}^{h-1}s\left[\frac{ks}{h}\right]=(h-1)(k-1)(8hk-h-k-1).\tag{1.8}$$

Thus

$$\begin{aligned}6kS_2(h,k)+6hS_2(k,h)&=(h-1)(k-1)(8hk-h-k-1)-(h^2-1)(k-1)(2k-1)\\&\quad-(k^2-1)(h-1)(2h-1)\end{aligned}$$

and (1.7) follows at once.

Incidentally, (1.8) is equivalent to the reciprocity theorem for Dedekind sums [3, p. 4]:

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{1}{hk}+\frac{k}{h}\right),\tag{1.9}$$

where

$$s(h,k)=\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{r}{k}-\left[\frac{r}{k}\right]-\frac{1}{2}\right).\tag{1.10}$$

2. Define

$$S_n(h,k)=\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^n\qquad(n=0,1,2,\cdots).\tag{2.1}$$

Thus $S_1(h,k)$ is evaluated by (1.1) while $S_2(h,k)$ satisfies (1.7). It is not difficult to show that a similar result holds for $S_3(h,k)$. We shall prove that

$$4k(k-1)S_3(h,k)+4h(h-1)S_3(k,h)=(h-1)^2(k-1)^2(2hk-h-k+1),\tag{2.2}$$

where of course $(h,k)=1$.

To prove (2.2), take

$$S_3(h,k) = \sum_{r=1}^{k-1} \left[\frac{h(k-r)}{k} \right]^3 = \sum_{r=1}^{k-1} \left[h - \frac{hr}{k} \right]^3 = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k} \right] \right)^3$$
$$= (h-1)^3(k-1) - 3(h-1)^2 S_1(h,k) + 3(h-1)S_2(h,k) - S_3(h,k) ,$$

so that, by (1.1),

(2.3) $$2S_3(h,k) = 3(h-1)S_2(h,k) - ½(h-1)^3(k-1).$$

Thus

$$4k(k-1)S_3(h,k) = 6(h-1)(k-1)kS_2(h,k) - (h-1)^3(k-1)^2k,$$

so that, by (1,7),

$$4k(k-1)S_3(h,k) + 4h(h-1)S_3(k,h) = (h-1)(k-1)\{6kS_2(h,k) + 6hS_2(k,h)\} - (h-1)^2(k-1)^2(2hk-h-k)$$
$$= (h-1)^2(2h-1)(k-1)^2(2k-1) - (h-1)^2(k-1)^2(2hk-h-k)$$
$$= (h-1)^2(k-1)^2 \{(2h-1)(2k-1) - (2hk-h-k)\}$$
$$= (h-1)^2(k-1)^2(2hk-h-k+1).$$

This proves (2.2).

If we apply the same method to $S_4(h,k)$, we get

$$S_4(h,k) = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k} \right] \right)^4$$
$$= (h-1)^4(k-1) - 4(h-1)^3 S_1(h,k) + 6(h-1)^2 S_2(h,k) - 4(h-1)S_3(h,k) + S_4(h,k),$$

which reduces to

$$4S_3(h,k) - 6(h-1)S_2(h,k) + (h-1)^3(k-1) = 0$$

in agreement with (2.3).

Generally, for arbitrary positive n,

$$S_n(k,k) = \sum_{r=1}^{k-1} \left(h - 1 - \left[\frac{hr}{k} \right] \right)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (h-1)^{n-j} S_j(h,k).$$

In particular, we have

$$S_{2n+1}(h,k) = \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} (h-1)^{2n-j+1} S_j(h,k),$$

so that

(2.4) $$2S_{2n+1}(h,k) = -½(2n-1)(h-1)^{2n+1}(k-1) + \sum_{j=2}^{2n} (-1)^j \binom{2n+1}{j} (h-1)^{2n-j+1} S_j(h,k).$$

Similarly,

$$S_{2n}(h,k) = \sum_{j=0}^{2n} (-1)^n \binom{2n}{j} (h-1)^{2n-j} S_j(h,k),$$

which reduces to

(2.5) $$-(n-1)(h-1)^{2n-1}(k-1) + \sum_{j=2}^{2n-1} (-1)^j \binom{2n}{j} (h-1)^{2n-j-1} S_j(h,k) = 0.$$

For example, for $n = 2$, (2.4) becomes

$$2S_5(h,k) = -\frac{3}{2}(h-1)^5(k-1) + 10(h-1)^3 S_2(h,k) - 10(h-1)^2 S_3(h,k) + 5S_4(h,k),$$

while, for $n = 3$, (2.5) becomes

$$-2(h-1)^5(k-1)+15(h-1)^3S_2(h,k)-20(h-1)^2S_3(h,k)$$
$$+15(h-1)S_4(h,k)-6S_5(h,k) = 0.$$

Combining these two formulas we get

$$\tfrac{1}{2}(h-1)^3(k-1)-3(h-1)S_2(h,k)+2S_3(h,k) = 0,$$

which is the same as (2.3).

It seems plausible that $S_n(h,k)$ satisfies a relation similar to (1.7) and (2.2) for every $n \geq 2$. However we are unable to prove this.

3. Consider the sum

$$\sum_{r=0}^{k-1} \phi^3\left(\frac{hr}{k}\right) = \sum_{r=0}^{k-1}\left(\frac{hr}{k}-\left[\frac{hr}{k}\right]\right)^3 = \sum_{r=1}^{k-1}\left(\frac{hr}{k}\right)^3 - 3\sum_{r=1}^{k-1}\left(\frac{hr}{k}\right)^2\left[\frac{hr}{k}\right] + 3\sum_{r=1}^{k-1}\frac{hr}{k}\left[\frac{hr}{k}\right]^2$$
$$-\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^3.$$

Now put

$$S_{i,j}(h,k) = \sum_{r=1}^{k-1} r^i\left[\frac{hr}{k}\right]^j, \qquad S_j = S_{0,j}. \tag{3.1}$$

Since

$$\sum_{r=0}^{k-1}\phi^3\left(\frac{hr}{k}\right) = \sum_{r=0}^{k-1}\phi^3\left(\frac{r}{k}\right) = \sum_{r=0}^{k-1}\left(\frac{r}{k}\right)^3 = \frac{1}{4k}(k-1)^2,$$

we get

$$S_3(h,k) - \frac{3h}{k}S_{1,2}(h,k) + \frac{3h^2}{k^2}S_{2,1}(h,k) - \frac{1}{4k}(k-1)^2(h^3-1) = 0. \tag{3.2}$$

In the next place,

$$S_{2,1}(h,k) = \sum_{r=1}^{k-1}(k-r)^2\left(h-1-\left[\frac{hr}{k}\right]\right) = \frac{1}{6}k(k-1)(2k-1)(h-1) - \sum_{r=1}^{k-1}(k-r)^2\left[\frac{hr}{k}\right]$$
$$= \frac{1}{6}k(k-1)(2k-1)(h-1) - \frac{1}{2}(h-1)(k-1)k^2 + 2kS_{1,1})h,k) - S_{2,1}(h,k),$$

so that

$$S_{2,1}(h,k) - kS_{1,1}(h,k) + \frac{1}{12}k(k-1)(2k-1)(h-1) - \frac{1}{4}(h-1)(k-1)k^2. \tag{3.3}$$

Similarly

$$S_{1,2}(h,k) = \sum_{r=1}^{k-1}(k-r)\left(h-1-\left[\frac{hr}{k}\right]\right)^2 = \tfrac{1}{2}k(k-1)(h-1)^2 - 2(h-1)\sum_{r=1}^{k-1}(k-r)\left[\frac{hr}{k}\right]$$
$$+\sum_{r=1}^{k-1}(k-r)\left[\frac{hr}{k}\right]^2 = \tfrac{1}{2}k(k-1)(h-1)^2 - k(k-1)(h-1)^2$$
$$+2(h-1)S_{1,1}(h,k) + kS_2(h,k) - S_{1,2}(h,k),$$

so that

$$S_{1,2}(h,k) = \tfrac{1}{2}kS_2(h,k) + (h-1)S_{1,1}(h,k) - \tfrac{1}{4}k(k-1)(h-1)^2. \tag{3.4}$$

By (1.8)

$$12hS_{1,1}(h,k) + 12kS_{1,1}(k,h) = (h-1)(k-1)(8hk-h-k-1).$$

Thus (3.3) yields

$$12h^2S_{2,1}(h,k) + 12k^2S_{2,1}(k,h) = hk(h-1)(k-1)(8hk-h-k-1) - 6h^2k^2(h-1)(k-1) + k(k-1)(2k-1)h^2(h-1) + h(h-1)(2h-1)k^2(k-1).$$

Simplifying, we get

$$12h^2S_{2,1}(h,k) + 12k^2S_{2,1}(k,h) = hk(h-1)(k-1)(6hk-2h-2k-1). \tag{3.5}$$

However, comparing (3.4) with (1.7) and (1.8), it does not seem likely that $S_{1,2}(h;k)$ satisfies any relation similar to (3.5).

4. We consider next the double sum

$$R(h_1, h_2; k) = \sum_{r,s=0}^{k-1} \left[\frac{h_1r+h_2s}{k}\right]^2 \qquad ((h_1h_2, k) = 1). \tag{4.1}$$

We have

$$\sum_{r,s=0}^{k-1} \phi^2\left(\frac{h_1r+h_2s}{k}\right) = \sum_{r,s=0}^{k-1}\left(\frac{h_1r+h_2s}{k} - \left[\frac{h_1r+h_2s}{k}\right]\right)^2 = \frac{1}{k^2}R_1 - \frac{2}{k}R_2 + R_3, \tag{4.2}$$

where

$$\begin{cases} R_1 = \sum_{r,s=0}^{k-1} (h_1r+h_2s)^2 \\ R_2 = \sum_{r,s=0}^{k-1} (h_1r+h_2s)\left[\frac{h_1r+h_2s}{k}\right] \\ R_3 = R(h_1, h_2, k). \end{cases}$$

Clearly

$$R_1 = \frac{1}{6}h_1^2k^2(k-1)(2k-1) + \frac{1}{2}h_1h_2k^2(k-1)^2 + \frac{1}{6}h_2^2k^2(k-1)(2k-1). \tag{4.3}$$

In the next place, by (1.2),

$$\sum_{r,s=0}^{k-1} r\left[\frac{h_1r+h_2s}{k}\right] = \sum_{r=0}^{k-1} r\left\{h_1r + \frac{1}{2}(h_2-1)(k-1)\right\} = \frac{1}{6}h_1k(k-1)(2k-1) + \frac{1}{4}(h_2-1)k(k-1)^2.$$

Similarly

$$\sum_{r,s=0}^{k-1} s\left[\frac{h_1r+h_2s}{k}\right] = \frac{1}{6}h_2k(k-1)(2k-1) + \frac{1}{4}(h_1-1)k(k-1)^2,$$

so that

$$R_2 = \frac{1}{6}h_1^2k(k-1)(2k-1) + \frac{1}{4}(2h_1h_2-h_1-h_2)k(k-1)^2 + \frac{1}{6}h_2^2k(k-1)(2k-1). \tag{4.4}$$

On the other hand, in view of (1.5),

$$\sum_{r,s=0}^{k-1} \phi^2\left(\frac{h_1r+h_2s}{k}\right) = \sum_{r,s=0}^{k-1}\phi^2\left(\frac{r+s}{k}\right) = k\sum_{t=0}^{k-1}\phi^2\left(\frac{t}{k}\right) = k\sum_{t=0}^{k-1}\left(\frac{t}{k}\right)^2 = \frac{1}{6}(k-1)(2k-1). \tag{4.5}$$

Hence, by (4.2), (4.3), (4.4), (4.5), we have

$$\frac{1}{6}(k-1)(2k-1) = \frac{1}{6}h_1^2(k-1)(2k-1) + \frac{1}{2}h_1h_2(k-1)^2 + \frac{1}{6}h_2^2(k-1)(2k-1)$$
$$- \frac{1}{3}h_1^2(k-1)(2k-1) - \frac{1}{2}(2h_1h_2-h_1-h_2)(k-1)^2 - \frac{1}{3}h_2^2(k-1)(2k-1) + R(h_1, h_2; k).$$

Simplifying, we get

(4.6) $$R(h_1, h_2; k) = \frac{1}{6}(h_1^2+h_2^2)(k-1)(2k-1) + \frac{1}{2}(h_1h_2 - h_1 - h_2)(k-1)^2 + \frac{1}{6}(k-1)(2k-1).$$

Next, put

(4.7) $$R(h_1, h_2, h_3; k) = \sum_{r,s,t=0}^{k-1} \left[\frac{h_1r+h_2s+h_3t}{k}\right]^2 \quad ((h_1h_2h_3, k) = 1).$$

Then, exactly as above,

$$\sum_{r,s,t=0}^{k-1} \phi^2\left[\frac{h_1r+h_2s+h_3t}{k}\right] = \sum_{r,s,t=0}^{k-1}\left(\frac{h_1r+h_2s+h_3t}{k} - \left[\frac{h_1r+h_2s+h_3t}{k}\right]\right)^2$$

$$= \frac{1}{k^2}R_1 - \frac{2}{k}R_2 + R_3,$$

where

$$R_1 = \sum_{r,s,t=0}^{k-1}(h_1r+h_2s+h_3t)^2, \qquad R_2 = \sum_{r,s,t=0}^{k-1}(h_1r+h_2s+h_3t)\left[\frac{h_1r+h_2s+h_3t}{k}\right],$$

$$R_3 = R(h_1, h_2, h_3; k).$$

Clearly

(4.8) $$R_1 = \frac{1}{6}k^3(k-1)(2k-1)\sum h_1^2 + \frac{1}{2}k^3(k-1)^2\sum h_1h_2,$$

where the sums on the right denote symmetric functions.

By (1.2),

$$\sum_{r,s,t=0}^{k-1} r\left[\frac{h_1r+h_2s+h_3t}{k}\right] = \sum_{r,s=0}^{k-1} r\left\{h_1r+h_2s+\frac{1}{2}(h_3-1)(k-1)\right\}$$

$$= \frac{1}{6}h_1k^2(k-1)(2k-1) - \frac{1}{4}(h_2+h_3-1)k^2(k-1)^2.$$

It follows that

(4.9) $$R_2 = \frac{1}{6}k^2(k-1)(2k-1)\sum h_1^2 + \frac{1}{4}\left(2\sum h_1h_2 - \sum h_1\right)k^2(k-1)^2.$$

Thus

$$\sum_{r,s,t=0}^{k-1}\phi^2\left(\frac{h_1r+h_2s+h_3t}{k}\right) = \frac{1}{6}k(k-1)(2k-1)\sum h_1^2 + \frac{1}{2}k(k-1)^2\sum h_1h_2$$

$$-2\left\{\frac{1}{6}k(k-1)(2k-1)\sum h_1^2 + \frac{1}{4}k(k-1)^2\left(2\sum h_1h_2 - \sum h_1\right)\right\} + R(h_1,h_2,h_3;k)$$

(4.10) $$= -\frac{1}{6}k(k-1)(2k-1)\sum h_1^2 - \frac{1}{2}k(k-1)^2\sum h_1h_2 + \frac{1}{2}k(k-1)^2\sum h_1$$

$$+ R(h_1, h_2, h_3; k).$$

On the other hand

$$\sum_{r,s,t=0}^{k-1}\phi^2\left(\frac{h_1r+h_2s+h_3t}{k}\right) = \sum_{r,s,t=0}^{k-1}\phi^2\left(\frac{r+s+t}{k}\right) = k^2\sum_{r=0}^{k-1}\phi^2\left(\frac{r}{k}\right) = \sum_{r=0}^{k-1}r^2 = \frac{1}{6}k(k-1)(2k-1).$$

Comparison with (4.10) gives

$$(4.11)\qquad R(h_1,h_2,h_3;k) = \frac{1}{6}k(k-1)(2k-1)\sum h_1^2 + \frac{1}{2}k(k-1)^2\sum h_1h_2$$

$$-\frac{1}{2}k(k-1)^2\sum h_1 = \frac{1}{6}k(k-1)(2k-1).$$

It will now be clear how to evaluate

$$(4.12)\qquad R(h_1,\cdots,h_n;k) = \sum_{r_i=0}^{k-1}\left[\frac{h_1r_1+\cdots+h_nr_n}{k}\right]^2 \qquad ((h_1h_2\cdots h_n,k)=1)$$

for any n.

The writer has proved [1] that the sum

$$(4.13)\qquad S(b,c;a) = \sum_{r,s=1}^{a-1} rs\left[\frac{br+cs}{a}\right]$$

satisfies

$$(4.14)\qquad S(1,b;a) = 4a^4b-10a^3b+a^4+8a^2b-2ab-a^2,$$

where $(a,b)=1$. The proof of (4.14) is rather complicated.

5. Summary. For the convenience of the reader, we restate the main results proved above.

$$(5.1)\qquad 6k\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^2 + 6h\sum_{s=1}^{h-1}\left[\frac{ks}{h}\right]^2 = (h-1)(2h-1)(k-1)(2k-1).$$

$$(5.2)\qquad 4k(k-1)\sum_{r=1}^{k-1}\left[\frac{hr}{k}\right]^3 + 4h(h-1)\sum_{s=1}^{h-1}\left[\frac{ks}{h}\right]^3$$

$$= (h-1)^2(k-1)^2(2hk-h-k+1).$$

$$(5.3)\qquad 12h^2\sum_{r=1}^{k-1} r^2\left[\frac{hr}{k}\right] + 12k^2\sum_{s=1}^{h-1} s^2\left[\frac{ks}{h}\right] = hk(h-1)(k-1)(6hk-2h-2k+1).$$

$$(5.4)\qquad \sum_{r,s=0}^{k-1}\left[\frac{h_1r+h_2s}{k}\right]^2 = \frac{1}{6}(h_1^2+h_2^2)(k-1)(2k-1)+\frac{1}{2}(h_1h_2-h_1-h_2)(k-1)^2+\frac{1}{6}(k-1)(2k-1).$$

$$(5.5)\qquad \sum_{r,s,t=0}^{k-1}\left[\frac{h_1r+h_2s+h_3t}{k}\right]^2 = \frac{1}{6}k(k-1)(2k-1)\left(\sum h_1^2+1\right)+\frac{1}{2}k(k-1)^2\left(\sum h_1h_2-\sum h_1\right).$$

In (5.1), (5.2), (5.3) it is assumed that $(h,k)=1$; in (5.4), $(h_1h_2,k)=1$; in (5.5), $(h_1h_2h_3,k)=1$.

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