ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman; 709 Solano Dr., S.E.; Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$, unless otherwise specified.

PROBLEMS PROPOSED IN THIS ISSUE

B-346 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Establish a closed form for

$$\sum_{k=1}^{n} F_{2k} T_{n-k} \quad + T_n + 1,$$

where T_k is the triangular number

$$\binom{k+2}{2} = (k+2)(k+1)/2.$$

B-347 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let a, b, and c be the roots of $x^3 - x^2 - x - 1 = 0$. Show that

$$\frac{a^n - b^n}{a - b} + \frac{b^n - c^n}{b - c} + \frac{c^n - a^n}{c - a}$$

is an integer for $n = 0, 1, 2, \cdots$.

B-348 Proposed by Sidney Kravitz, Dover, New Jersey.

Let $P_1, \cdots, P_5$ be the vertices of a regular pentagon and let Q_i be the intersection of segments $\overline{P_{i+1}P_{i+3}}$ and $\overline{P_{i+2}P_{i+4}}$ (subscripts taken modulo 5). Find the ratio of lengths $\overline{Q_1Q_2}/\overline{P_1P_2}$.

B-349 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots$ be the sequence 1, 1, 2, 2, 3, 3, $\cdots$, i.e., let a_n be the greatest integer in $1 + (n/2)$. Give a recursion formula for the a_n and express the generating function

$$\sum_{n=0}^{\infty} a_n x^n$$

as a quotient of polynomials.

B-350 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let a_n be as in B-349. Find a closed form for

$$\sum_{k=0}^{n} a_{n-k}(a_k + k)$$

in the case (a) in which n is even and the case (b) in which n is odd.

B-351 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Prove that $F_4 = 3$ is the only Fibonacci number that is a prime congruent to 3 modulo 4.

SOLUTIONS

FRONT PAGE ALPHAMETIC

B-322 Proposed by Sidney Kravitz, Dover, New Jersey.

Solve the following alphametic in which no 6 appears:

$$\begin{array}{r} \text{A R K I N} \\ \text{A L D E R} \\ \underline{\text{S A L L E}} \\ \text{A L L A D I} \end{array}$$

(All the names are taken from the front cover of the April, 1975 *Fibonacci Quarterly.*)

Solution by Charles W. Trigg, San Diego, California.

$A = 1$, whereupon $R + 2 = 10$, so $R = 8$. Then $S + 3 = L + 10$. This has two possible solutions: $S = 7$, $L =$ zero and $S = 9$, $L = 2$.

If $S = 7$, $L =$ zero, the subsequent values follow immediately, namely: $N = 4$, $E = 3$, $I = 5$, $D = 9$, and $K = 2$. Thus the reconstructed addition is

$$18254 + 10938 + 71003 = 100195 .$$

Also solved by Richard Blazej, John W. Milsom, C. B. A. Peck, and the Proposer.

VARIATIONS ON AN OLD THEME

B-323 Proposed by J. A. H. Hunter, Fun with Figures, Toronto, Ontario, Canada.

Prove that
$$F_{n+r}^2 - (-1)^r F_n^2 = F_r F_{2n+r} .$$

Solution by George Berzsenyi, Lamar University, Beaumont, Texas.

The identity is a restatement of I_{19} of Hoggatt's *Fibonacci and Lucas Numbers* with (k,n) replaced by $(n,n+r)$. It may be proven directly by using the Binet-formulas:

$$\begin{aligned} F_{n+r}^2 - (-1)^r F_n^2 &= \left(\frac{a^{n+r} - b^{n+r}}{a-b}\right)^2 - (-1)^r \left(\frac{a^n - b^n}{a-b}\right)^2 \\ &= \frac{1}{(a-b)^2}\,[a^{2n+2r} + b^{2n+2r} - 2(ab)^{n+r} - (-1)^r(a^{2n} + b^{2n} - 2(ab)^n)] \\ &= \frac{1}{(a-b)^2}\,[a^{2n+2r} + b^{2n+2r} - (-1)^r b^{2n} - (-1)^r a^{2n}] \\ &= \frac{1}{(a-b)^2}\,[a^{2n+2r} + b^{2n+2r} - (ab)^r b^{2n} - (ab)^r a^{2n}] \\ &= \frac{a^r - b^r}{a-b}\,\frac{a^{2n+r} - b^{2n+r}}{a-b} = F_r F_{2n+r} . \end{aligned}$$

Also solved by Richard Blazej, Wray G. Brady, Herta T. Freitag, Ralph Garfield, Frank Higgins, Graham Lord, John W. Milsom, Carl F. Moore, C. B. A. Peck, Bob Prielipp, J. Shallit, Sahib Singh, Gregory Wulczyn, and the Proposer.

FIBONACCI CONGRUENCE

B-324 Proposed by Herta T. Freitag, Roanoke, Virginia.

Determine a constant k such that, for all positive integers n,

$$F_{3n+2} \equiv k^n F_{n-1} \pmod 5.$$

Solution by Graham Lord, Université Laval, Québec, Canada.

$$F_{3n+2} = F_6 F_{3n-3} + F_5 F_{3n-4} = F_6 \cdot F_{n-1} \cdot [5F_{n-1}^2 + 3(-1)^{n-1}] + 5F_{3n-4}$$
$$\equiv (-1)^n F_{n-1} \pmod 5.$$

Also solved by George Berzsenyi, Ralph Garfield, Frank Higgins, Bob Prielipp, J. Shallit, Sahib Singh, Gregory Wulczyn, and the Proposer.

IMPOSSIBLE FUNCTIONAL EQUATION

B-325 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Prove that there does not exist an even single-valued function G such that

$$x + G(x^2) = G(ax) + G(bx) \quad \text{on} \quad -a \leqslant x \leqslant a.$$

Solution by Graham Lord, Université Laval, Québec, Canada.

There does not exist a single-valued function G which satisfies the equation since if $x = a$, one finds that $a = G(ab)$ and for $x = b$ that $b = G(ab)$; the two results together violate the single-valuedness. (Note that G need not be even.)

Also solved by George Berzsenyi, Wray G. Brady, Frank Higgins, C. B. A. Peck, and the Proposer.

ON THE SUM OF DIVISORS

B-326 Based on the Solution to B-303 by David Zeitlin, Minneapolis, Minnesota.

For positive integers n, let $\sigma(n)$ be the sum of the positive integral divisors of n. Prove that

$$\sigma(mn) > 2\sqrt{\sigma(m)\sigma(n)} \quad \text{for} \quad m > 1 \quad \text{and} \quad n > 1.$$

Solution by Bob Prielipp, The University of Wisconsin, Oshkosh, Wisconsin.

In B-260 it was shown that $\sigma(mn) > \sigma(m) + \sigma(n)$ for $m > 1$ and $n > 1$. By the arithmetic mean—geometric mean inequality, $\sigma(m) + \sigma(n) \geqslant 2\sqrt{\sigma(m)\sigma(n)}$. The desired result follows immediately.

Also solved by Herta T. Freitag, Frank Higgins, Graham Lord, Carl F. Moore, J. Shallit and Sahib Singh.

FINISHING TOUCHES ON A LUCAS IDENTITY

B-327 Proposed by George Berzsenyi, Lamar University, Beaumont, Texas.

Find all integral values of r and s for which the equality

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{ri} = s^n L_n$$

holds for all positive integers n.

Solution by Frank Higgins, Naperville, Illinois.

For $n = 1$ and $n = 2$ we obtain the equations $2 - L_r = s$ and $2 - 2L_r + L_{2r} = 3s^2$, respectively. Replacing s by $2 - L_r$ in the second equation we have $L_{2r} = 10 - 10L_r + 3L_r^2$ which, since $L_{2r} = L_r^2 - 2(-1)^r$, reduced to $(L_r - 2)(L_r - 3) = 0$ for r even and to $(L_r - 1)(L_r - 4) = 0$ for r odd. Thus $r = 0, 1, 2, 3$ and $s = 2 - L_r = 0, 1, -1, -2$, respectively, are the only possible pairs of solutions. We now show that each pair is, in fact, a solution for all positive integers n. Using the Binet form we have

$$s^n(\alpha^n+\beta^n) = s^nL_n = \sum_{i=0}^{n}\binom{n}{i}(-1)^iL_{ri} = \sum_{i=0}^{n}\binom{n}{i}(-1)^i(\alpha^r)^i$$

$$+\sum_{i=0}^{n}\binom{n}{i}(-1)^i(\beta^r)^i = (1-\alpha^r)^n+(1-\beta^r)^n$$

from which it is readily verified that $r = 0, 1, 2, 3$ and $5 = 0, 1, -1, -2$, respectively, are solutions.

Also solved by Herta T. Freitag, Ralph Garfield, and the Proposer.

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ADVANCED PROBLEMS AND SOLUTIONS

$$\begin{cases} \alpha^2A(n)+\alpha B(n) = 0 \\ (\alpha-\beta)C(n) = 0 \\ A(n)+\alpha B(n) = 0 \end{cases}$$

It follows at once that

$$A(n) = B(n) = C(n) = 0 \qquad (n \geqslant 0).$$

It is evident that a similar result holds for the Lucas numbers and similar sequences of numbers.

Also solved by P. Tracy and P. Bruckman.

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