A REARRANGEMENT OF SERIES BASED ON A PARTITION OF THE NATURAL NUMBERS

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Good [3] showed that

$$(1)\qquad \sum_{m=0}^{n} \frac{1}{F_{2^m}} = 3 - \frac{F_{2^n-1}}{F_{2^n}}, \qquad n \geqslant 1.$$

The problem of summing this for $n \to \infty$ was posed by Millin [8]. The bibliography at the end of this paper gives an idea of what has been done with such series and their extensions. A common thread may be found among many of these studies: explicit or implicit use is made of an interesting partition of the natural numbers. Our object here will be to discuss this partition and generalize it, as well as show other uses. Our main results are some series rearrangement formulas that are related to multi-sections but differ and do not seem to appear in the literature.

Our first observation is that the set $\{(2k+1)2^n \mid k \geqslant 0,\ n \geqslant 0\}$ is identical to the set of all natural numbers. Holding either k or n fixed and letting the other variable assume all non-negative integers, we find that the natural numbers are generated as the union of countably many disjoint subsets of the naturals. Pictorially, every natural number appears once and only once in the array:

1	3	5	7	9	11	13	15	17	19	...
2	6	10	14	18	22	26	...			
4	12	20	28	36	...					
8	24	40	...							
16	48	...								
32	96	...								

This seems to be common knowledge in the mathematical community, but its use in forming interesting series rearrangements does not seem to be widely known or appreciated. The rearrangement theorem is as follows:

$$(2)\qquad \sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f((2k+1)2^n)$$

for an arbitrary function f provided only that the series on the left converges absolutely so that it can be rearranged at will. For a convergent series of positive terms, of course, the formula always holds. The theorem is used by Greig [4] to obtain the transformation

$$(3)\qquad \sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{\substack{k=1\\ k\text{ odd}}}^{\infty} \left\{C_k - \frac{1}{a}\right\}, \qquad a = \frac{1+\sqrt{5}}{2},$$

where

$$(4)\qquad C_k = \begin{cases} (1+F_{k-1})/F_k & \text{for even } k, \\ (1+F_{k-1})/F_k + 2/F_{2k} & \text{for odd } k. \end{cases}$$

The numbers C_k arose in his proof that (1) generalizes to

$$\sum_{m=0}^{n} \frac{1}{F_{k2^m}} = C_k - \frac{F_{k2^n-1}}{F_{k2^n}}, \qquad k,n \geqslant 1, \tag{5}$$

but he did not make explicit use of (2) in determining (5), the numbers C_k being introduced in the course of an inductive proof.

On the other hand, according to Hoggatt and Bicknell [5, p. 275, Method X], Carlitz used what is essentially (2) to sum (1) when $n \to \infty$. To make this as clear as possible, we rephrase the argument as follows: With a, b the roots of $z^2 - z - 1 = 0$, so that $ab = -1$, and $a - b = \sqrt{5}$, then the Binet formula is $F_n = (a^n - b^n)/(a - b)$, and so

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \sum_{n=1}^{\infty} \frac{a-b}{a^{2^n} - b^{2^n}} = (a-b) \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} a^{-(2k+1)2^n}$$

$$= (a-b) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a^{-(2k+1)2^n} - \sum_{k=0}^{\infty} a^{-2k-1},$$

and the double series can be summed by using (2), so that the result follows since everything is then known by simple geometric sums.

If we apply the same argument to the Lucas numbers, recalling that $L_n = a^n + b^n$, we find that

$$\sum_{n=1}^{\infty} \frac{1}{L_{2^n}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k a^{-(2k+1)2^n}, \tag{6}$$

but the presence of the factor $(-1)^k$ prevents us from going further as (2) cannot be applied then. Perhaps some other result can be found using (6).

The formula

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{x}{1-x}, \qquad |x| < 1, \tag{7}$$

attributed by Bromwich [1, p. 24] to Augustus De Morgan follows easily out of (2): For $|x| < 1$,

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{(2k+1)2^n} = \sum_{n=0}^{\infty} x^{2^n} \sum_{k=0}^{\infty} (x^{2^{n+1}})^k$$

$$= \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}},$$

and this is substantially the way that many related results can be found.

For instance, either using (7) or going back to (2) again, we may set down the hyperbolic trigonometric analogue of (1) which is done for $n \to \infty$ in (22) below.

We come now to the generalization of (2). Going first to mod 3, we have:

$$\sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((3k+1)3^n) + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((3k+2)3^n), \tag{8}$$

provided only that the series on the left converges absolutely.

The two disjoint sets

$$\{(3k+1)3^n \mid k \geqslant 0,\ n \geqslant 0\} \quad \text{and} \quad \{(3k+2)3^n \mid k \geqslant 0,\ n \geqslant 0\}$$

form an interesting partition of the natural numbers. The two sets are easily put down in the arrays

1	4	7	10	13	16	⋯
3	12	21	30	39	48	⋯
9	36	63	90	⋯		
27	108	⋯				
81	324	⋯				

and

2	5	8	11	14	17	⋯
6	15	24	33	42	51	⋯
18	45	72	99	⋯		
54	135	⋯				
162	405	⋯				

The general case, mod m, is:

$$\sum_{n=1}^{\infty} f(n) = \sum_{i=1}^{m-1} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f((mk+i)m^n), \qquad m \geqslant 2, \tag{9}$$

provided the series on the left converges absolutely.

We should remark that when $f(n)$ is replaced by $f(n)x^n$ we may use (9) and its special cases as a theorem on formal power series and matters of convergence may be ignored when we use such a formula to equate coefficients in proving combinatorial formulae. Tutte [9] has given an interesting new theory of formal power series.

Formula (9) may be further generalized usefully. It is not difficult to see that multiples of powers of m may be removed from the set of natural numbers and we obtain the following nice result:

$$\sum_{i=1}^{m-1} \sum_{k=0}^{\infty} \sum_{n=0}^{r} f((mk+i)m^n) = \sum_{n=1}^{\infty} f(n) - \sum_{n=1}^{\infty} f(m^{r+1}n), \quad m \geqslant 2,\ r \geqslant 0, \tag{10}$$

$$= \sum_{n=0}^{\infty} f(n) - \sum_{n=0}^{\infty} f(m^{r+1}n), \quad m \geqslant 2,\ r \geqslant 0,$$

provided that the series converge absolutely. Notice that the series on the right may be written in an alternative manner when $f(0)$ is defined as then the first terms cancel out. This allows us often to write a more elegant formula.

We pause now to exhibit a neat application of (10) to derive a general formula found by Bruckman and Good [2] whose argument is tantamount to formula (10) but it was not explicitly stated. We have, with $f(n) = x^n$,

$$\sum_{i=1}^{m-1} \sum_{k=0}^{\infty} \sum_{n=0}^{r} x^{(mk+i)m^n} = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} x^{m^{r+1}n}$$

so that

$$\frac{1}{1-x} - \frac{1}{1-x^{m^{r+1}}} = \sum_{n=0}^{r} \sum_{i=1}^{m-1} x^{im^n} \sum_{k=0}^{\infty} x^{m^{n+1}k}$$

$$= \sum_{n=0}^{r} \sum_{i=1}^{m-1} x^{im^n} (1 - x^{m^{n+1}})^{-1} = \sum_{n=0}^{r} \frac{1 - x^{m^n(m-1)}}{(1-x^{m^n})(1-x^{m^{n+1}})} x^{m^n}, \tag{11}$$

which proves the finite series result in [2]. This formula, of course, is the extension to values other than $m = 2$ of De Morgan's formula (7) and in a finite setting.

We pause to exhibit a non-Fibonacci application of (10). For the Riemann Zeta function we find

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{i=1}^{m-1} \sum_{k=0}^{\infty} \frac{1}{(mk+i)^s} \sum_{n=0}^{\infty} \frac{1}{m^{sn}}, \qquad s > 1,$$

which simplifies to

$$\left(1-\frac{1}{m^s}\right)\zeta(s)=\sum_{i=1}^{m-1}\sum_{k=0}^{\infty}\frac{1}{(mk+i)^s}, \qquad s>1, \tag{12}$$

or

$$(m^s-1)\zeta(s)=\sum_{i=1}^{m-1}\zeta(s,i/m), \tag{13}$$

in terms of Hurwitz' generalized Zeta function, which is defined by

$$\zeta(s,a)=\sum_{k=0}^{\infty}\frac{1}{(k+a)^s}, \qquad s>1,\ a \text{ arbitrary},$$

so that $\zeta(s,1)=\zeta(s)$. But formula (12) or (13) is not new. It is the same result found by using ordinary multisection modulo m.

Ordinary multisection means the following formula:

$$\sum_{n=1}^{\infty} f(n)=\sum_{i=1}^{m}\sum_{k=0}^{\infty} f(mk+i), \qquad m\geqslant 1, \tag{14}$$

the result again being valid for absolutely convergent series on the left.

Since we are speaking of multisection, it may be worthwhile to set down the formula corresponding to (14) for a finite series:

$$\sum_{k=a}^{n} f(k)=\sum_{i=0}^{m-1}\sum_{k=\left[\frac{a+m-1-i}{m}\right]}^{\left[\frac{n-i}{m}\right]} f(mk+i), \qquad n-a+1\geqslant m\geqslant 1 \tag{15}$$

where brackets denote the usual greatest integer function.

Finite multisection in the form (15) has always been a favorite of the author, and it has two interesting further special cases worth setting down for reference:

$$\sum_{k=0}^{mn-1} f(k)=\sum_{i=0}^{m-1}\sum_{k=0}^{n-1} f(mk+i), \qquad m\geqslant 1,\ n\geqslant 1; \tag{16}$$

and

$$\sum_{k=0}^{mn} f(k)=\sum_{i=1}^{m}\sum_{k=0}^{n-1} f(mk+i), \qquad m\geqslant 1,\ n\geqslant 0. \tag{17}$$

It is well known that there is an analogy between the formulas for Fibonacci-Lucas numbers and trigonometric functions. To every formula involving Fibonacci and Lucas numbers there is a corresponding formula involving sines and cosines. We know that this is true because of the similarities between the Binet formulas

$$F_n=\frac{a^n-b^n}{a-b}, \qquad L_n=a^n+b^n \tag{18}$$

and the Euler formulas

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i}, \qquad \cos x=\frac{e^{ix}+e^{-ix}}{2}, \qquad i^2=-1. \tag{19}$$

The same may be said for the hyperbolic functions:

$$\sinh x=\frac{e^x-e^{-x}}{2}, \qquad \cosh x=\frac{e^x+e^{-x}}{2}, \tag{20}$$

and we merely cite, e.g., relations like $\sin 2x = 2\sin x \cos x$, $\sinh 2x = 2\sinh x \cosh x$, $F_{2n} = F_n L_n$ to remind of the analogy. It is natural then to set down trigonometric analogues of formulas we have discussed above.

The case $n \to \infty$ of (1) was

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2} = 2.381966012\cdots, \tag{21}$$

and the hyperbolic sine analogue is

$$\sum_{n=0}^{\infty} \frac{1}{\sinh 2^n} = \frac{2}{e-1} = 1.163953414\cdots. \tag{22}$$

When $n \to \infty$ in (5) the special case of Greig's formula is

$$\sum_{n=0}^{\infty} \frac{1}{F_{k2^n}} = C_k - \frac{\sqrt{5}-1}{2}, \qquad k \geqslant 1, \tag{23}$$

C_k being given by (4), and the hyperbolic analogue is

$$\sum_{n=0}^{\infty} \frac{1}{\sinh 2^n x} = \frac{2}{e^x - 1} \qquad x > 0. \tag{24}$$

Although (7) and its congeners are often listed in compendia of series, I am not aware of any ready listing for them written in the hyperbolic form (24), not even (22).

Possibilities exist for application to number theoretic functions. Since g.c.d. $(mk+i, m^n) = 1$ for all $1 \leqslant i \leqslant m-1$, we may apply (2), (8), (9), (10) to multiplicative number theoretic functions as well as completely multiplicative functions. For instance, using Euler's ϕ-function, we find from (2),

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\phi(2k+1)\phi(2^n)}{(2k+1)^s 2^{ns}} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\phi(2k+1)\phi(2^n)}{(2k+1)^s 2^{ns}} + \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s} \tag{25}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{ns-n+1}} \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s} + \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s}, \qquad s > 2,$$

which I have not seen stated elsewhere. Since we can also use ordinary multisection of series we have besides

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\phi(2n)}{(2n)^s} + \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s}, \qquad s > 2, \tag{26}$$

whence, upon comparing (25) and (26) we get the unusual formula

$$\sum_{n=1}^{\infty} \frac{1}{2^{ns-n+1}} \sum_{k=0}^{\infty} \frac{\phi(2k+1)}{(2k+1)^s} = \sum_{n=1}^{\infty} \frac{\phi(2n)}{(2n)^s}, \qquad s > 2. \tag{27}$$

To get these results we used $\phi(p^n) = p^n - p^{n-1}$ (p = any prime), and similar formulas to (25) and (27) may be found for other multiplicative functions. A more complicated result follows with $f = \phi$ in (9) or (10).

We should note that (27) is exactly analogous to the formula

$$\sum_{n=1}^{\infty} \frac{a^{2n}}{a^{4n}-1} = \sum_{\substack{k=1\\ k\ \text{odd}}}^{\infty} \frac{1}{a^{2k}-1}, \qquad a = \frac{1+\sqrt{5}}{2}, \tag{28}$$

which was found in Greig's paper [4] by an entirely analogous procedure, and which I do not believe is immediately obvious.

Besides these applications it is clear that the general formulas we have given, (2), (8), (9), (10), may be applied with success to the many generalizations of the Fibonacci-Lucas sequence that have been studied. It is hoped that our remarks may shed some light on the nature of the formula (1) and its analogues and why others fail to exist. For example, what can be said about (22) with sine instead of hyperbolic sine?

A final observation is that our formulas sometimes give transformed series that are very rapidly convergent. Thus (10) gives

$$\sum_{n=1}^{\infty} f(n) = \sum_{k=0}^{\infty} \sum_{n=0}^{r-1} f((2k+1)2^n) + \sum_{n=1}^{\infty} f(2^r n), \tag{29}$$

and when we can sum the double series, we may take a very large but convenient r and expect the remaining infinite series to converge very rapidly. Thus, for the Fibonacci case, using Greig's formulas, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sum_{n=0}^{r-1} \left\{ \frac{1+F_{2n}}{F_{2n+1}} + \frac{2}{F_{4n+2}} - \frac{1}{a} \right\} + \sum_{n=1}^{\infty} \frac{1}{F_{2^r n}}. \tag{30}$$

For $r = 10$, 20, or 100 we could sum the first part and the remaining infinite series needs only a few terms to get a good approximation. I suppose this is an old trick but I am not able to cite a reference. The method must have been used before.

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