AN APPLICATION OF W. SCHMIDT'S THEOREM TRANSCENDENTAL NUMBERS AND GOLDEN NUMBER

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INTRODUCTION

Recently, W. Schmidt proved the following theorem.

Schmidt's Theorem. Let 1, a_1, a_2 be algebraic real numbers, linearly independent over $\mathbb{Q}$, and let $\epsilon > 0$. There are only finitely many integers q such that

(1) $$\|qa_1\|\,\|qa_2\| \leqslant c_1 q^{-1-\epsilon},$$

where c_1 is a positive constant and where $\|\ \|$ denotes the distance from the nearest integer.

Of course, this theorem can be used to prove that certain numbers are transcendental. We shall take a_1 equal to the golden number. The integers q will be chosen in the sequence of Fibonacci numbers. It remains only to take a number a_2 such that $\|qa_2\|$ is small for these values of q and such that 1, a_1, a_2 are $\mathbb{Q}$-linearly independent. We shall give only one example of such a number a_2 but the proof shows clearly that there are many other possible choices of a_2.

THE RESULT

Proposition. Let $(u_1, u_2, \cdots)$ be the sequence of Fibonacci numbers. Put $q_n = u_{2^n}$. Then the number

$$a_2 = \sum_{n=1}^{\infty} \frac{2+(-1)^n}{q_n}$$

is transcendental.

Proof. It is well known and easily proved that

$$\left| a_1 - \frac{u_{n+1}}{u_n} \right| \sim \frac{1}{\sqrt{5}\,u^2}$$

Thus,

(2) $$\|q_n a_1\| \sim \frac{1}{\sqrt{5}\,q_n}.$$

Since $u_n | u_{2n}$, q_n divides q_{n+1}. Hence,

$$\sum_{n=1}^{N} \frac{2+(-1)^n}{q_n} = \frac{p_n}{q_n},$$

where p_n is an integer.

Now, it is easily proved that

$$\left| a_2 - \frac{p_N}{q_N} \right| \sim \frac{2+(-1)^{N+1}}{q_{N+1}} \sim \frac{(2+(-1)^{N+1})\sqrt{5}}{q_N^2}.$$

Thus,

(3) $$\|q_N a_2\| \sim \frac{(2+(-1)^{N+1})\sqrt{5}}{q_N}.$$

From (2) and (3), we get

$$\|q_N a_1\| \, \|q_N a_2\| \leq \frac{c}{q_N^2},$$

where c is a positive constant.

We have verified that (1) holds with $\epsilon = 1$.

It remains only to show that $1, a_1, a_2$ are linearly independent over $\mathbb{Q}$. Suppose that we can find a non-trivial relation

$$k_0 + k_1 a_1 + k_2 a_2 = 0, \qquad k_i \in \mathbb{Q}.$$

We can now limit ourselves to the case of $k_i \in \mathbf{Z}$. For large N, the previous relation gives

$$k_1 \|q_N a_1\| = \pm k_2 \|q_N a_2\| .$$

This contradicts (2) and (3). Thus, $1, a_1, a_2$ are $\mathbb{Q}$-linearly independent. Now Schmidt's theorem shows that a_2 is not algebraic. The assertion is proved.

REMARK. The proposition remains true if we put

$$u_n = \frac{x^n - y^n}{x - y},$$

where x is a quadratic Pisot number and y its conjugate.

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For small integers n the positive solutions of (1) may be found with a machine because of the upper bound of n^2 on the coordinates. For $n = 3$ these solutions are exactly those revealed in the general case. That is, (3,3,3) and permutations of (1,2,3).

In the complementary case (that is, some coordinate is negative), there are, for each $n > 1$, always an infinite number of solutions. For example, $(a, 1, -a)$, for any integer a, satisfies (1) in case $n = 3$. For $n = 4$, $(a, a, -a; -a)$ satisfies (1), etc. For $n = 3$ the solution will be a subset of the solutions of

$$x_1^3 + x_2^3 + x_3^3 = u^2,$$

an identified problem [1, p. 566].

In case $n = 2$ the reader will have no difficulty in showing that all solutions are $(a, -a)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ together, of course, with $(0, 0)$, $(0, 1)$, $(1, 0)$ which come from the case $n = 1$. The case $n = 2$ is a special case of a well known theorem [1, p. 412 *et seq.*].

REFERENCE

1. L. E. Dickson, *History of the Theorey of Numbers,* Vol. II, Carnegie Institution of Washington, D.C., 1920.

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