ON THE N CANONICAL FIBONACCI REPRESENTATIONS OF ORDER N

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SUMMARY

Carlitz, Scoville and Hoggatt [1, 2] have investigated Fibonacci representations of higher order. In this paper we introduce for each $N \geqslant 2$ a series of N distinct canonical Fibonacci representations of order N for each positive integer n which we call the first canonical through the N^{th} canonical representations. The first canonical representation parallels the usual Zeckendorff representation and the N^{th} canonical representation parallels what the aforementioned authors have called the second canonical representation. For each of these canonical representations there is determined a table W_N^k analogous to the tables studied in [1, 3]. For $0 < k < N$ the tables W_N^k are shown to be tables of Fibonacci differences of order k of the columns of W_N^0, which is the table generated by the first canonical representation. As a result we obtain a remarkable theorem which states that for every $0 < k < N$ the table of Fibonacci differences of order k of the columns of W_N^0 inherits the following characteristics (and more) from the table W_N^0: (1) Every entry of the table is a positive integer and every positive integer occurs exactly once as an entry in the table and, (2) Every row and every column of the table is increasing. It is interesting to note that no such table exists with analogous properties in terms of ordinary differences even for $N = 3$. In the latter part of the paper we give a generating function for the canonical sequences (those which generate the canonical representations) and also give the extension of the elegant procedure in [1, 3] for generating the tables W_2^0 and W_3^0.

1. THE N CANONICAL REPRESENTATION OF ORDER N

A sequence $\{G_i\}_{i=1}^{\infty}$ shall be called a *Fibonacci sequence of order* N $(N \geqslant 2)$ iff

$$\sum_{j=0}^{N-1} G_{i+j} = G_{i+N} \quad \text{for every} \quad i = 1, 2, \cdots.$$

The particular Fibonacci sequence $\{F_{N,i}\} = \{F_{N,i}^0\}$ of order N determined by the initial conditions $F_{N,i} = 2^{i-1}$, $i = 1, 2, \cdots, N$ is called the sequence of *Fibonacci numbers of order* N.* For each integer $k = 1, 2, \cdots, N-1$ we define a Fibonacci sequence $\{F_{N,i}^k\}$ of order N by

$$F_{N,i}^k = F_{N,i+k} - \sum_{j=0}^{k-1} F_{N,i+j}, \qquad i = 1, 2, 3, \cdots.$$

Given a Fibonacci sequence $\{G_i\}$ of order N and a positive integer n, a *canonical representation of* n *by the sequence* $\{G_i\}$ is a sum

$$n = \sum k_i G_i$$

in which (i) the summation extends over all positive indices i and all but a finite number of the k_i are zero, (ii) $k_i \neq 0 \Rightarrow k_i = 1$ and

(iii) $$\prod_{j=0}^{N-1} k_{i+j} = 0 \qquad \text{for all } i$$

*This enumeration of the Fibonacci numbers is shifted by one from that in [1, 2, 3]; this shifting seems to be indicated by Theorem 1.1.

The largest index i such that $k_i \neq 0$ is called the *upper degree* of the representation and the smallest index i such that $k_i \neq 0$ is called the *lower degree* of the representation. The principal result of this section is the following theorem

Theorem 1.1. Let N be a fixed integer greater than one and let k be any integer between 0 and $N-1$, inclusive. Then every positive integer n has one and only one canonical representation by $\{F_{N,i}^k\}$ of lower degree congruent to one of the integers $\{1, 2, \cdots, N-k\}$ modulo N.

Note that for $k = 0$ the theorem gives uniqueness of canonical representations by $\{F_{N,i}\}$ without restricting the lower degree of the representation. At the other extreme, canonical representations with respect to $\{F_{N,i}^{N-1}\}$ are required to have lower degree congruent to 1 modulo N. This, combined with the observation that

$$F_{N,i} = F_{N,i+1}^{N-1} \quad \text{for} \quad i = 1, 2, 3, \cdots$$

explains the connection of these representations by $\{F_{N,i}^{N-1}\}$ with the representations in [1, 3] called second canonical.

For each $k = 1, 2, \cdots, N$, the unique representation by $\{F_{N,i}^{k-1}\}$ guaranteed by Theorem 1.1 shall be called the k^{th} *canonical Fibonacci representation of order N.*

The proof of Theorem 1.1 is accomplished with the aid of four lemmas.

Lemma 1.1. Let $\{G_i\}$ be a Fibonacci sequence of order N which is non-decreasing and satisfies $G_1 = 1$ and $G_{i+1} \leqslant 2G_i$ for all i. Then for every positive integer n, a canonical representation of n by $\{G_i\}$ can be obtained from the following algorithm, which we shall call *exhaustion.* Let G_{i_1} be the term of $\{G_i\}$ of largest index satisfying $G_i \leqslant n$. If $G_{i_1} \neq n$ let G_{i_2} be the term of $\{G_i\}$ of largest index satisfying $G_i \leqslant n - G_{i_1}$. Continue inductively; after finitely many steps an index i_p will be found such that

$$n = \sum_{j=1}^{p} G_{i_j},$$

and this sum will be a canonical representation of n by $\{G_i\}$.

Proof. Because $G_1 = 1$ and because $\{G_i\}$ must be unbounded, each term of the sequence $i_1, i_2, \cdots, i_p$, as well as p itself, is well defined. From $2G_i \geqslant G_{i+1}$ we must have $i_1 > i_2 > \cdots > i_p$ since the equality of any adjacent pair of these indices would contradict the choice of the one with smaller subscript. If there exist among $i_1, i_2, \cdots, i_p$ sets of N consecutive integers, let $i_k, i_{k+1}, \cdots, i_{k+N-1}$ be that set having first index i_k of smallest subscript k. Then

$$\sum_{j=k}^{k+N-1} G_{i_j} = G_{i_k+1}$$

which contradicts the choice of i_k.

Lemma 1.2. Let $\{G_i\}$ be a positive term Fibonacci sequence of order N having the property that

$$\sum_{i=1}^{k} G_i \leqslant G_{k+1} \quad \text{for} \quad k = 1, 2, \cdots, N-1.$$

Then (i) $\{G_i\}$ is strictly increasing except possibly for $G_1 = G_2$ and (ii) if $\sum k_i G_i$ is any canonical representation by $\{G_i\}$ and if the upper degree of representation is p, then $\sum k_i G_i \leqslant G_{p+1}$.

Proof. The validity of (i) is clear as is that of (ii) for $1 \leqslant p \leqslant N$. Suppose (ii) holds for all $p < m$ for some $m > N$. Of all sums determined by canonical representations by $\{G_i\}$ of upper degree m let n be the largest. If n is represented canonically by $\{G_i\}$, each of the numbers $G_m, G_{m-1}, \cdots, G_{m-N+2}$ must be present in the representation since otherwise its sum could be increased without altering its canonical properties or its upper degree. The number G_{m-N+1} cannot be present, and so by the same reasoning G_{m-N} must be present unless it happens that $m - N = 2$ and $G_2 = G_1$, in which case G_1 must be present if G_2 is not and can be replaced by G_2 without altering the sum. It then follows that

$$n - \sum_{i=m-N+2}^{m} G_i$$

has a canonical representation by $\{G_i\}$ of upper degree $m-N$, which by the inductive hypothesis cannot sum to more than G_{m-N+1}, so

$$n \leqslant \sum_{i=m-N+1}^{m} G_i = G_{m+1}.$$

Given a Fibonacci sequence $\{G_i\}$ of order N, a term G_i shall be called *redundant* if G_i can be expressed as a sum of fewer than N terms of distinct subscripts from among $\{G_j, G_{j+1}, \cdots, G_{i-1}\}$, where $j = \max\{1, i-N\}$. We shall make use of the observation that a positive term Fibonacci sequence of order N can contain no redundant terms beyond the first N.

Lemma 1.3. Let $\{G_i\}$ be a Fibonacci sequence of order N which satisfies the hypothesis of Lemma 1.2. Suppose some positive integer n has two distinct canonical representations by $\{G_i\}$. Then $\{G_i\}$ has a redundant term G_r for which one of the two canonical representations of n has lower degree congruent to r modulo N.

Proof. Proof is by induction on the maximum p of the upper degrees of the two representations of n. The case $p = 1$ is vacuous. Suppose the lemma holds for all $p < m$ and let $p = m$. If both representations have upper degree m, subtract G_m from both and apply the inductive hypothesis. Otherwise by Lemma 1.2 the representation of smaller upper degree can sum to at most G_m so the representation having upper degree m must consist of the single term G_m. If $m \leqslant N$ then G_m is redundant and $r = m$. If $m > N$ the other representation must have upper degree $m-1$ by Lemma 1.2, and must contain all of the numbers $G_{m-1}, G_{m-2}, \cdots, G_{m-N+1}$ since otherwise its value could be increased beyond that of G_m in contradiction to Lemma 1.2. Since it is canonical it cannot contain the number G_{m-N}. Therefore, upon removal of the terms $G_{m-1}, G_{m-2}, \cdots, G_{m-N+1}$ from the representation there results a canonical representation for

$$n - \sum_{i=m-N+1}^{m-1} G_i = G_{m-N}$$

with upper degree less than $m-N$. By the inductive hypothesis either the lower degree of this representation for G_{m-N} is congruent to r modulo N, in which case the same is true of the canonical representation from which it was derived by the removal of the last $N-1$ terms of the latter, or else $m-N$ is congruent to r modulo N, in which case the same is true of the lower degree of the other representation $n = G_m$.

Lemma 1.4. Let N be an integer greater than one and let k be a nonnegative integer less than N. Then the redundant terms of $\{F^k_{N,i}\}$ are precisely $F^k_{N,N-k+1}, F^k_{N,N-k+2}, \cdots, F^k_{N,N}$, and in fact

$$F^k_{N,i} = \sum_{j=1}^{i-1} F^k_{N,j}, \qquad i = N-k+1, \cdots, N.$$

We note that $\{F^0_{N,i}\}$ has no redundant terms.

Proof. By definition $F_{N,i} = 2^{i-1}$ for $i = 1, 2, \cdots, N$. By summation we obtain $F_{N,N+1} = 2^N - 1$ which proves for $i = 1$ the formula

$$F_{N,N+i} = 2^{i-2}(2^{N+1} - i - 1), \qquad i = 1, 2, \cdots, N.$$

Proof for $2 \leqslant i \leqslant N$ follows by induction, using the relation $F_{N,N+i} = 2F_{N,N+i-1} - F_{N,i-1}$. By direct calculation one now finds that $F^k_{N,i} = 2^{i-1}$ for $i = 1, 2, \cdots, N-k$, so that none of these terms can be redundant. Again by direct calculation one finds that

$$F^k_{N,N-k+1} = 2^{N-k} - 1 = \sum_{i=1}^{N-k} F^k_{N,i}$$

which verifies the statement of the lemma for $i = N - k + 1$. Suppose the lemma is true for $i < N - k + j$ for some j such that $1 < j \leqslant k$. Then for $i = N - k + j$ we have $N + 1 < i + k \leqslant 2N$ so that

$$F^k_{N,i} = F_{N,i+k} - \sum_{p=0}^{k-1} F_{N,i+p} = 2F_{N,i+k-1} - F_{N,i+k-1-N} - \sum_{p=0}^{k-1} F_{N,i+p}$$

$$= F_{N,i+k-1} - F_{N,i+k-1-N} - \sum_{p=1}^{k-1} F_{N,i+p-1} = F^k_{N,i-1} + F_{N,i-1} - F_{N,i+k-1-N}$$

$$= \sum_{p=1}^{i-2} F^k_{N,p} + F_{N,i-1} - F_{N,i+k-1-N}$$

by the inductive hypothesis. But

$$F^k_{N,i-1} = F_{N,i+k-1} - \sum_{j=0}^{k-1} F_{N,i+j-1} = \sum_{p=i+k-N-1}^{i+k-2} F_{N,p} - \sum_{p=i-1}^{i+k-2} F_{N,p} = \sum_{p=i-(N-k+1)}^{i-2} F_{N,p}$$

since $i + k - 1 > N$. Since $i \leqslant N$ we have

$$\sum_{p=i-(N-k+1)}^{i-2} F_{N,p} = \sum_{p=i-(N-k+1)}^{i-2} 2^{p-1} = 2^{i-2} - 2^{i-(N-k+2)} = F_{N,i-1} - F_{N,i+k-1-N}\,.$$

This gives

$$\sum_{p=1}^{i-2} F^k_{N,p} + F_{N,i-1} - F_{N,i+k-1-N} = \sum_{p=1}^{i-1} F^k_{N,p}$$

and the induction is complete.

Proof of Theorem 1.1. By the information contained in the statement and proof of Lemma 1.4 we see that

$$F^k_{N,i} = 2^{i-1} \quad \text{for} \quad i = 1, 2, \cdots, N-k,$$

that

$$F^k_{N,N-k+1} = 2F^k_{N,N-k} - 1 \quad \text{and that} \quad F^k_{N,i+1} = 2F^k_{N,i} \quad \text{for} \quad N-k+1 < i \leqslant N,$$

the latter following from

$$F^k_{N,i+1} = \sum_{j=1}^{i} F^k_{N,j} = F^k_{N,i} + \sum_{j=1}^{i-1} F^k_{N,j} = 2F^k_{N,i}\,.$$

For $k = 0$ we know that $F^k_{N,N+1} = 2F^k_{N,N} - 1$ and for $k = 1, 2, \cdots, N - 1$ we have, as above,

$$F^k_{N,N+1} = \sum_{i=1}^{N} F^k_{N,i} = 2F^k_{N,N}\,.$$

Thus for each $\{F^k_{N,i}\}$ we have

$$1 = F^k_{N,1} \leqslant F^k_{N,2} \leqslant \cdots \leqslant F^k_{N,N} \quad \text{and} \quad F^k_{N,i+1} \leqslant 2F^k_{N,i} \quad \text{for } i = 1, 2, \cdots, N+1.$$

It now follows by induction that $\{F^k_{N,i}\}$ satisfies the hypothesis of Lemma 1.1, and it is clear that $\{F^k_{N,i}\}$ satisfies the hypothesis of Lemma 1.2 and hence also of Lemma 1.3. By Lemma 1.1 each positive integer has by exhaustion a canonical representation by $\{F^k_{N,i}\}$. This representation fails to satisfy the condition imposed by the theorem on the lower degree only if it has lower degree of the form $mN + p$, $N - k < p \leqslant N$. For this case we describe a method for obtaining a canonical representation of the desired form which we shall call *reduction.* Replace

$$F^k_{N,mN+p} \quad \text{by} \quad \sum_{i=(m-1)N+p}^{mN+p-1} F^k_{N,i}$$

and then replace

$$F^k_{N,(m-1)N+p} \quad \text{by} \quad \sum_{i=(m-2)N+p}^{(m-1)N+p-1} F^k_{N,i},$$

and so on, until arriving at

$$\sum_{i=p}^{N+p-1} F^k_{N,i}.$$

According to Lemma 1.4 we can now replace

$$F^k_{N,p} \quad \text{by} \quad \sum_{i=1}^{p-1} F^k_{N,i},$$

and the end result of all these replacements is seen to be a $k+1^{st}$ canonical representation by $\{F^k_{N,i}\}$ of lower degree one. The uniqueness of this representation comes immediately from Lemmas 1.3 and 1.4.

Given a Fibonacci sequence $\{G_i\}$ of order N and a system of canonical representations by $\{G_i\}$, we shall say that the system is *lexicographic* if whenever

$$m = \sum k_i G_i \quad \text{and} \quad n = \sum k_i' G_i$$

are two canonical representations in the system, then $m < n$ if and only if the representations

$$\sum k_i G_i \quad \text{and} \quad \sum k_i' G_i$$

differ and differ in such a way that the largest i for which $k_i \neq k_i'$ has $k_i = 0$, $k_i' = 1$. Clearly this property implies uniqueness within the system (although it does not imply existence within the system or uniqueness outside of the system).

Theorem 1.2. For each $N \geqslant 2$ and for each nonnegative $k < N$ any system of unique representations by $\{F^k_{N,i}\}$ is lexicographic.

Proof. This theorem is an extension of Lemma 1.2. Suppose that

$$\sum k_i F^k_{N,i} \quad \text{and} \quad \sum k_i' F^k_{N,i}$$

differ and that $k_p = 0$, $k_p' = 1$ and $k_i = k_i'$ for all $i > p$. Then remove

$$\sum_{i>p} k_i F^k_{N,i} = \sum_{i>p} k_i' F^k_{N,i}$$

from both representations, so that it is sufficient to show that

$$\sum_{i=1}^{p} k_i F^k_{N,i} < \sum_{i=1}^{p} k_i' F^k_{N,i}.$$

Since the upper degree of

$$\sum_{i=1}^{p} k_i F^k_{N,i}$$

is less than p, by Lemma 1.2 the sum cannot exceed

$$F_{N,p}^k \leqslant \sum_{i=1}^{p} k_i' F_{N,i}^k .$$

Thus we have

$$\sum_{i=1}^{p} k_i F_{N,i}^k < \sum_{i=1}^{p} k_i' F_{N,i}^k ,$$

since if the two sums were equal one could replace

$$\sum_{i>p} k_i F_{N,i}^k = \sum_{i>p} k_i' F_{N,i}^k$$

and contradict the uniqueness assumption.

Suppose that m and n are positive integers having canonical representations within the system and that $m < n$. Let the canonical representations be

$$m = \sum k_i F_{N,i}^k \quad \text{and} \quad n = \sum k_i' F_{N,i}^k .$$

By the uniqueness of canonical representations within the system, the only way the theorem can fail is for these two representations to differ with $k_p = 1$, $k_p' = 0$ and $k_i = k_i'$ for all $i > p$ which gives $m > n$ by the first half of the theorem.

Theorem 1.3. Let $N \geqslant 2$ and $1 \leqslant k < N$. Then no positive integer has more than two distinct canonical representations by $\{F_{N,i}^k\}$. A number has two distinct canonical representations by $\{F_{N,i}^k\}$ if and only if the representation given by exhaustion* is not $k + 1^{st}$ canonical, and therefore all canonical representations by $\{F_{N,i}^k\}$ can be found by first applying exhaustion* and then (if the result is not $k + 1^{st}$ canonical) reduction**.

Proof. It suffices to prove that if a positive integer n has two distinct canonical representations by $\{F_{N,i}^k\}$, then the one which is lexicographically inferior is $k + 1^{st}$ canonical and the other is given by exhaustion. Let

$$n = \sum k_i F_{N,i}^k = \sum k_i' F_{N,i}^k$$

canonically with the first representation lexicographically inferior. Let $k_p = 0$, $k_p' = 1$, $k_i = k_i'$ for all $i > p$, so that

$$\sum_{i \leqslant p-1} k_i F_{N,i}^k = \sum_{i \leqslant p} k_i' F_{N,i}^k .$$

By Lemma 1.2

$$\sum_{i \leqslant p-1} k_i F_{N,i}^k = \sum_{i \leqslant p} k_i' F_{N,i}^k = F_{N,p}^k .$$

If $k < N - 1$, the representation

$$\sum_{i \leqslant p-1} k_i F_{N,i}^k$$

and thus also the representation

$$\sum k_i F_{N,i}^k$$

*Defined in the statement of Lemma 1.1.
**Defined in the proof of Theorem 1.1.

must be $k+1^{st}$ canonical since otherwise

$$F_{N,1}^k + \sum_{i \le p-1} k_i F_{N,i}^k$$

is also canonical and exceeds $F_{N,p}^k$, in contradiction to Lemma 1.2. For $k = N-1$ the same remarks apply unless $k_1 = 0$ and $k_2 = k_3 = \cdots = k_N = 1$, which cannot happen, since if it did we would have

$$\sum_{i=2}^{p-1} k_i F_{N,i}^{N-1} = \sum_{i=1}^{p-2} k_{i+1} F_{N,i}^0 = F_{N,p}^{N-1} = F_{N,p-1}^0$$

contradicting the uniqueness of the first canonical representation.

It remains only to show that

$$\sum k_i' F_{N,i}^k$$

is given by exhaustion. If it were not, it would be lexicographically inferior to that representation of

$$\sum k_i' F_{N,i}^k$$

which was given by exhaustion, which by what has already been proven would make

$$\sum k_i' F_{N,i}^k$$

$k+1^{st}$ canonical.

2. THE TABLES W_N^k AND FIBONACCI DIFFERENCES

We now fix $N \ge 2$ and fix k such that $0 \le k < N$ and consider the set of $k+1^{st}$ canonical representations. For each $i = 1, 2, \cdots, N-k$ let $\{a_{i,j}^k\}_{j=1}^\infty$ be the sequence generated by listing in increasing order those positive integers having $k+1^{st}$ canonical representations with lower degree congruent to i modulo N, and denote the $(N-k)$-rowed infinite matrix $((a_{i,j}^k))$ by W_N^k. W_2^0 and W_3^0 have been discussed by Carlitz, Scoville and Hoggatt [1, 3].

The following theorem is an immediate consequence of the lexicographic property of the $k+1^{st}$ canonical representation.

Theorem 2.1. If the $k+1^{st}$ canonical Fibonacci representation of order N of $a_{1,j}^k$ is

$$\sum k_p F_{N,p}^k,$$

then for $a_{i,j}^k$ it is

$$\sum k_p F_{N,p+i-1}^k, \qquad i = 2, 3, \cdots, N-k.$$

The $k+1^{st}$ canonical Fibonacci representation of order N for $a_{i,j}^k$ and the first canonical representation for $a_{i,j}^0$ have identical coefficient sequences $\{k_p\}$.

Corollary. Each matrix W_N^k has the following properties:

(1) Every entry of W_N^k is a positive integer and every positive integer occurs exactly once as an entry of W_N^k,

(2) Every row and every column of W_N^k is increasing,

(3) For $k = 1, 2, \cdots, N-1$, for any $i,j \le N-k$ and for any p,q, $a_{i,p}^k < a_{j,q}^k$ if and only if $a_{i,p}^0 < a_{j,q}^0$, and

(4) $a_{i+1,j}^k \le 2a_{i,j}^k$ for $i = 1, 2, \cdots, N-k-1$.

Statement (4) makes use of the property $F_{N,i+1}^k \le 2F_{N,i}^k$, $i = 1, 2, \cdots$, verified in the proof of Theorem 1.1. Another useful corollary is the following.

Corollary. Let n be a positive integer. Then if the N^{th} canonical representation of n is

$$\sum k_p F_{N,p}^{N-1},$$

the $k+1^{st}$ canonical representation of $a_{i,n}^{k}$ is

$$\sum k_p F_{N,p+i-1}^{k}, \qquad i = 1, 2, \cdots, N-k, \quad k = 0, 1, \cdots, N-1.$$

Proof. By Theorem 2.1, the first canonical representations of $a_{1,n}^{0}$ and the N^{th} canonical representation of $a_{1,n}^{N-1}$ have identical coefficient sets. But by statements (1) and (2) of the preceding corollary and the fact that W_N^{N-1} has just one row, we see that $a_{1,n}^{N-1} = n$ for every positive integer n. Thus if the N^{th} canonical representation of n is

$$\sum k_p F_{N,p}^{N-1},$$

the first canonical representation of $a_{1,n}^{0}$ is

$$\sum k_p F_{N,p}^{0}$$

and so by Theorem 2.1 the $k+1^{st}$ canonical representation of $a_{1,n}^{k}$ is

$$\sum k_p F_{N,p}^{k}$$

and that of $a_{i,n}^{k}$ is

$$\sum k_p F_{N,p+i-1}^{k}.$$

Given an N-tuple $(a_1, a_2, \cdots, a_N)$ and given an integer $k = 1, 2, \cdots, N-1$, we define an $(N-k)$-tuple called the k^{th} *Fibonacci difference of* $(a_1, a_2, \cdots, a_N)$ by

$$\phi^k(a_1, a_2, \cdots, a_N) = (b_1, b_2, \cdots, b_{N-k})$$

with

$$b_i = a_{i+k} - \sum_{j=0}^{k-1} a_{i+j}, \qquad i = 1, 2, \cdots, N-k.$$

Then we can prove the following theorem.

Theorem 2.2. For each $N \geqslant 2$ and for each $k = 1, 2, \cdots, N-1$, every column of W_N^k is the k^{th} Fibonacci difference of its corresponding column in W_N^0. Thus the tables of k^{th} Fibonacci differences of the columns of W_N^0 enjoy all of the properties listed in the first corollary to the preceding theorem.

Proof. By Theorem 2.1 we have

$$a_{i,j}^{k} = \sum k_p F_{N,p+i-1}^{k},$$

where

$$\sum k_p F_{N,p}^{k}$$

is the $k+1^{st}$ canonical representation of $a_{1,j}^{k}$. But

$$F_{N,p+i-1}^{k} = F_{N,p+i+k-1} - \sum_{r=0}^{k-1} F_{N,p+i+r-1}$$

which gives

$$a_{i,j}^k = \sum_p k_p F_{N,p+i+k-1} - \sum_{r=0}^{k-1} \sum_p k_p F_{N,p+i+r-1}\,.$$

Again using Theorem 2.1 we obtain

$$a_{i,j}^k = a_{i,j+k}^0 - \sum_{r=0}^{k-1} a_{i,j+r}^0$$

which is the i,j entry of the table of k^{th} Fibonacci differences of the columns of W_N^0.

In Figure 2.1 we show a portion of W_4^0 with its accompanying tables of Fibonacci differences. One can see that the properties of the Fibonacci differences given in Theorem 2.2 suffice to determine the table of W_N^0 if it is also required that the rows of W_N^0 be increasing sequences forming a disjoint partition of the positive integers. If one tries the same thing for ordinary differences for $N = 3$, the result is shown in Fig. 2.2, wherein duplications occur in the third and fifth, fourth and seventh and fifth and ninth columns (as far as the table goes).

1	3	5	7	9	11	13	15	16	18	20
2	6	10	14	17	21	25	29	31	35	39
4	12	19	27	33	41	48	56	60	68	75
8	23	37	52	64	79	93	108	116	131	145

1	3	5	7	8	10	12	14	15	17	19
2	6	9	13	16	20	23	27	29	33	36
4	11	18	25	31	38	45	52	56	63	70

1	3	4	6	7	9	10	12	13	15	16
2	5	8	11	14	17	20	23	25	28	31

1	2	3	4	5	6	7	8	9	10	11

Fig. 2.1 A Portion of W_4^0 and Accompanying Fibonacci Difference Tables

1	3	5	7	8	10	12	14	15
2	6	9	13	16	19	23	26	29
4	11	16	23	29	34	41	46	52

1	3	4	6	8	9	11	12	14
2	5	7	10	13	15	18	20	23

1	2	3	4	5	6	7	8	9

Fig. 2.2 Counter-Example to Theorem 2.2 for Ordinary Differences $(N = 3)$

Our next theorem gives the generalization of the procedure used in [1, 3] to define W_2^0 and W_3^0.

Theorem 2.3. for each $N \geqslant 2$ and each $k = 0, 1, \cdots, N-2$,

$$a_{i+1,j}^k = 1 + a_{1,a_{i,j}^0}^k, \qquad i = 1, 2, \cdots, N-k-1.$$

We note that the information in this theorem is sufficient for the construction of W_N^0 in the sense of [1, 3], but not for the construction of W_N^k, unless W_N^0 has already been constructed.

Proof. A representation for $a^k_{1,a^0_{i,j}}$ can be obtained through the second corollary to Theorem 2.1 as follows. Let $a^k_{i+1,j}$ have $k+1^{st}$ canonical representation

$$\sum k_p F^k_{N,p}\,;$$

which therefore has lower degree congruent to $i+1$ modulo N. Then by Theorem 2.1 the first canonical representation for $a^0_{i,j}$ is

$$\sum k_p F^0_{N,p-1}\,.$$

Since $F^{N-1}_{N,p+1} = F^0_{N,p}$ for all p,

$$\sum k_p F^{N-1}_{N,p}$$

is a canonical representation for $a^0_{i,j}$ by $\{F^{N-1}_{N,i}\}$ which, however, is not N^{th} canonical because it has lower degree congruent to $i+1$ modulo N. By Theorem 1.3 the N^{th} canonical representation now follows by reduction. Let the lower degree of

$$\sum k_p F^{N-1}_{N,p}$$

be $mN+i+1$. Then by the nature of the reduction process we know that the N^{th} canonical representation of $a^0_{i,j}$ is given by

$$\sum_{p=1}^{i} F^{N-1}_{N,p} + \sum_{q=0}^{m-1}\sum_{r=i+2}^{N+i} F^{N-1}_{N,qN+r} + \sum_{p>mN+i+1} k_p F^{N-1}_{N,p}\,.$$

By the second corollary to Theorem 2.1 we have that the $k+1^{st}$ canonical representation of $a^k_{1,a^0_{i,j}}$ is

$$\sum_{p=1}^{i} F^{k}_{N,p} + \sum_{q=0}^{m-1}\sum_{r=i+2}^{N+i} F^{k}_{N,qN+r} + \sum_{p>mN+i+1} k_p F^{k}_{N,p}\,.$$

Now since

$$\sum k_p F^k_{N,p}$$

is a $k+1^{st}$ canonical representation of lower degree congruent to $i+1$ modulo N and with $i+1$ among the residues $0, 1, \cdots, N-k$ we must have $i < N-k$ and therefore

$$\sum_{p=1}^{i} F^k_{N,p} = F^k_{N,i+1} - 1$$

by what has been shown in the proof of Theorem 1.1. Thus if 1 is added to the $k+1^{st}$ canonical representation of $a^k_{1,a^0_{i,j}}$ the terms produced by the reduction process exactly recombine to yield the expression

$$\sum_p k_p F^k_{N,p}\,,$$

and hence

$$a^k_{i+1,j} = 1 + a^k_{1,a^0_{i,j}}\,.$$

Our last theorem provides a generating function for the sequences $\{F^k_{N,i}\}$.

Theorem 2.4. Let α be a (positive) root of

[Continued on page 34.]