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Also, since α and β satisfy (4), we have the equations

$$\alpha^{n+2} = \alpha^{n+1} + \left(\frac{p-1}{4}\right)\alpha^n, \qquad \beta^{n+2} = \beta^{n+1} + \left(\frac{p-1}{4}\right)\beta^n \qquad (n \geqslant 1).$$

Therefore, using (3), it follows that

$$G_{n+2} = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{p}} = \frac{\alpha^{n+1} + \left(\frac{p-1}{4}\right)\alpha^n - \beta^{n+1} - \left(\frac{p-1}{4}\right)\beta^n}{\sqrt{p}}$$
$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{p}} + \left(\frac{p-1}{4}\right)\frac{\alpha^n - \beta^n}{\sqrt{p}} = G_{n+1} + \left(\frac{p-1}{4}\right) G_n .$$

Thanks to (5) it is now a simple matter (despite the complicated appearance of (2)) to generate terms of the sequence $\{G_n\}$, for any choice of p. Assuming that we are interested only in integer-valued sequences, (5) tells us to take p of the form $4k+1$; namely $p = 1, 5, 9, 13, 17, \cdots$. Thus the first five such sequences start as follows:

p	$\frac{p-1}{4}$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	$\cdots$
1	0	1	1	1	1	1	1	1	1	1	1	$\cdots$
5	1	1	1	2	3	5	8	13	21	34	55	$\cdots$
9	2	1	1	3	5	11	21	43	85	171	341	$\cdots$
13	3	1	1	4	7	19	40	97	217	508	1159	$\cdots$
17	4	1	1	5	9	29	65	181	441	1165	2929	$\cdots$

We can use the above table to guess at various properties of the generalized Fibonacci sequence $\{G_n\}$, especially if our knowledge of $\{F_n\}$ is taken into account. Generalizations of some of the better-known properties of $\{F_n\}$ are listed below. Of course, in each case, the original result may be found by taking

$$p = 5, \qquad \frac{p-1}{4} = 1 \quad \text{and} \quad G_n = F_n .$$

(i) $$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \frac{1+\sqrt{p}}{2}$$

(ii) $$G_n \cdot G_{n+2} - G_{n+1}^2 = (-1)^{n+1}\left(\frac{p-1}{4}\right)^n \qquad (n \geqslant 1)$$

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