BIJECTION IN $Z^+ \times Z^+$

B-333 Proposed by Phil Mana, Albuquerque, New Mexico.

Let S_n be the set of ordered pairs of integers (a,b) with both $0 < a < b$ and $a + b \leqslant n$. Let T_n be the set of ordered pairs of integers (c,d) with both $0 < c < d < n$ and $c + d > n$. For $n \geqslant 3$, establish at least one bijection (i.e., 1-to-1 correspondence) between S_n and T_{n+1}.

I. Solution by Herta T. Freitag, Roanoke, Virginia; Frank Higgins, Naperville, Illinois; and the Proposer (each separately).

$$c = b \quad \text{and} \quad d = n + 1 - a$$

or inversely,

$$a = n + 1 - d \quad \text{and} \quad b = c.$$

II. Solution by Mike Hoffman, Warner Robins, Georgia; and the Proposer (separately).

$$c = n + 1 - b \quad \text{and} \quad d = n + 1 - a$$

or, inversely,

$$a = n + 1 - d \quad \text{and} \quad b = n + 1 - c.$$

It is straightforward to verify that $a + b \leqslant n$ if and only if $c + d > n$ and hence that each of I and II gives a one-to-one correspondence.

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ADVANCED PROBLEMS AND SOLUTIONS

$$= \frac{x^{\beta+1}w^{-n}}{(1-\beta)x+\beta} \sum_{j=0}^{n} \binom{n}{j} (1 - x^{\beta-1}w)^{-2j} \sum_{m=0}^{\infty} (-1)^{n+j+m} \binom{j}{m} (x^{\beta-1}w)^m (1 + x^{\beta-1}w)^j$$

$$= \frac{x^{\beta+1}(-w)^{-n}}{(1-\beta)x+\beta} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(\frac{1 + x^{\beta-1}w}{1 - x^{\beta-1}w}\right)^j = \frac{x^{\beta+1}(-w)^{-n}}{((1-\beta)x+\beta)} \left(\frac{-2x^{\beta-1}w}{1 - x^{\beta-1}w}\right)^n$$

$$= \frac{x^{\beta+1}2^n}{((1-\beta)x+\beta)} \left(\frac{x^{\beta-1}}{1 - x^{\beta-1}w}\right)^n = \frac{2^n x^{\beta n+\beta+1}}{(1-\beta)x+\beta} \quad .$$

Comparing this with (1), it is clear that we have proved the identity.

CORRECTION

H-267 (Corrected)

Show that

$$S(x) = \sum_{n=0}^{\infty} \frac{(kn+1)^{n-1}X^n}{n!}$$

satisfies $S(x) = e^{xS^k(x)}$.

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