ON A PROPERTY OF CONSECUTIVE FAREY-FIBONACCI FRACTIONS

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Krishnaswami Alladi [1] defined the Farey sequence of Fibonacci numbers of order F_n (where F_n is the n^{th} Fibonacci number) as the set of all possible fractions F_i/F_j, $i = 0, 1, \cdots, n-1; j = 1, 2, \cdots, n; (i < j)$ arranged in ascending order of magnitude, with the last item $1 (= F_1/F_2)$ and the first term $0 (= F_0/F_{n-1})$.

Now, the necessary and sufficient condition that the fractions h/k, h'/k', of $\mathcal{F}_n$, the n^{th} ordinary Farey section, be consecutive is that

(1) $$|kh' - hk'| = 1$$

and the fraction

(2) $$(h+h')/(k+k')$$

is not in $\mathcal{F}_n$.

All terms in $\mathcal{F}_{n+1}$ which are not in $\mathcal{F}_n$ are of the form $(h+h')/(k+k')$, where h/k and h'/k' are consecutive terms of $\mathcal{F}_n$. (Proofs of these results are given in Hardy and Wright [3].)

The usefulness of this result in the description of continued fractions in terms of Farey sections (Mack [5]) is an incentive to determine its Fibonacci analogue. (Also relevant are Alladi [2] and Mack [4].)

In the notation of Alladi where $f{\cdot}f_n$ denotes a Farey sequence of order F_n, the analogue of (2) above is:

All terms of $f{\cdot}f_{n+1}$ which are not already in $f{\cdot}f_n$ are of the form $(F_i + F_j)/(F_k + F_{k+1})$ where F_i/F_k and F_j/F_{k+1} are consecutive terms of $f{\cdot}f_n$ (with the exception of the first term which equals $0/F_n$).

The result follows from Alladi's definition of "generating fractions" and it can be illustrated by

$$f{\cdot}f_5: \quad 0/3,\ 1/5,\ 1/3,\ 2/5,\ 1/2,\ 3/5,\ 2/3,\ 1/1$$

and

$$f{\cdot}f_6: \quad 0/5,\ 1/8,\ 1/5,\ 2/8,\ 1/3,\ 3/8,\ 2/5,\ 1/2,\ 3/3,\ 5/8,\ 2/3,\ 1/1;$$

the terms of $f{\cdot}f_6$ which are not in $f{\cdot}f_5$ are

$$\frac{0}{5}, \frac{1}{8} = \frac{0+1}{3+5}, \quad \frac{2}{8} = \frac{1+1}{3+5}, \quad \frac{3}{8} = \frac{1+2}{3+5}, \quad \frac{5}{8} = \frac{3+2}{5+3}.$$

It is of interest to consider the analogue of (1) and here we have a result similar to Theorem 2.3 of Alladi [1]. Our problem is the following:

If $f_{(r)n} = h/k$, and $f_{(r+1)n} = h/k$ then to find $kh' - hk'$ purely in terms of r and n. We have the following theorem to this effect.

Theorem: Let $f_{(r)n} = h/k$ and $f_{(r+1)n} = h'/k'$. Then

$$kh' - hk' = \begin{cases} F_{n-1} & \text{for } r = 1 \\ F_{n-m} & \text{for } 1 < r \leqslant (n^2 - 7n + 14)/2 \\ 1 & \text{for } r > (n^2 - 7n + 14)/2 \end{cases},$$

where

$$m = 2 + [(\sqrt{8r-15} - 1)/2]$$

in which $[\cdot]$ is the greatest integer function.

Proof. The theorem follows if we combine Theorems 2.3 and 3.1a of Alladi [1]. By Theorem 2.3, if h/k and h'/k' are consecutive in $f \cdot f_n$ and satisfy

$$(3) \qquad \frac{1}{F_i} \leqslant \frac{h}{k} < \frac{h'}{k'} \leqslant \frac{1}{F_{i-1}}$$

then

$$(4) \qquad h' - h' = F_{i-2}\,.$$

So we first need to find the position of $1/F_i$ in $f \cdot f_n$. By Theorem 3.1a, if $f_{(r)n} = 1/F_{n-m}$ then

$$(5) \qquad r = 2 + \{1 + 2 + 3 + \cdots + \}\ .$$

So by (3) and (4) if $f_{(r)n} = h/k$, and $f_{(r+1)} = h'/k'$ then

$$kh' - hk' = F_{n-m}$$

if and only if

$$(6) \qquad \frac{1}{F_{n-m+2}} \leqslant f_{(r)n} \leqslant f_{(r+1)n} \leqslant \frac{1}{F_{n-m+1}}\ .$$

Now (6) and (5) combine to give

$$(7) \quad 2 + \{1 + 2 + \cdots + m - 2\} = \frac{m^2 - 3m + 6}{2} \leqslant r < r + 1 \leqslant 2 + \{1 + 2 + \cdots + m - 1\} = \frac{m^2 - m + 4}{2}\ .$$

Now the first inequality of (7) is essentially

$$(8) \qquad \begin{gathered} m^2 - 3m + 6 \leqslant 2r \iff \left(m - \frac{3}{2}\right)^2 + \frac{15}{4} \leqslant 2r \iff (2m - 3)^2 + 15 \leqslant 8r \\ \iff m \leqslant 2 + \frac{\sqrt{8r - 15} - 1}{2} = \frac{\sqrt{8r - 15} + 3}{2}\,. \end{gathered}$$

Similarly the second inequality in (7) may be expressed as

$$(9) \qquad \begin{gathered} r + 1 \leqslant \frac{m^2 - m + 4}{2} \iff r \leqslant \frac{m^2 - m + 2}{2} \iff 2r \leqslant (m - \tfrac{1}{2})^2 + \frac{7}{4} \\ \iff 8r \leqslant (2m - 1)^2 + 7 \iff \frac{\sqrt{8r - 7} + 1}{2} \leqslant m\,. \end{gathered}$$

Now consider for $r \geqslant 2$

$$(10) \qquad 0 < \frac{\sqrt{8r - 15} + 3}{2} - \frac{\sqrt{8r - 7} + 1}{2} = \frac{2 + \sqrt{8r - 15} - \sqrt{8r - 1}}{2} < 1\,.$$

Now (10), (9) and (8) together imply

$$m = \left[\frac{\sqrt{8r - 15} + 3}{2}\right] = 2 + \left[\frac{\sqrt{8r - 15} - 1}{2}\right]$$

and that proves the theorem for $r \geq 2$. For $r = 1$, the first statement is trivially true.

Since it is of interest if $kh' - hk' = 1$, let us determine when this occurs. This will happen if and only if (by (6) and (4))

$$(11) \qquad \frac{1}{F_4} \leqslant f_{(r)n}\ .$$

By (5) and (11) we have

$$r \geqslant 2 + \{1 + 2 + \cdots + n - 4\} = \frac{n^2 - 7n + 16}{2}$$

which is for

$$r > \frac{n^2 - 7n + 14}{2}$$

and that completes the proof.

REMARK. Note, in our theorem, if $f_{(r)n} = h/k$, and $f_{(r+1)n} = h'/k'$, we need not know the values of h/k, and h'/k' to determine $kh' - hk'$. This is determined purely in terms of r and n.

REFERENCES

1. Krishnaswami Alladi, "A Farey Sequence of Fibonacci numbers," *The Fibonacci Quarterly,* Vol. 13, No. 1, (Feb. 1975), pp. 1–10.
2. Krishnaswami Alladi, "Approximation of Irrationals with Farey Fibonacci Fractions," *The Fibonacci Quarterly,* Vol. 13, No. 3 (Oct. 1975), pp. 255–259.
3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Clarendon Press, Oxford, 1965, Ch. III.
4. J. M. Mack, "A Note on Simultaneous Approximation," *Bull. Austral. Math. Soc.,* Vol. 3 (1970), pp. 81-83.
5. J.M. Mack, "On the Continued Fraction Algorithm," *Bull. Austral. Math. Soc.,* Vol. 3 (1970), pp. 413–422.

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SUMS OF PRODUCTS INVOLVING FIBONACCI SEQUENCES

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Definition. $\{H_n\}$ is Fibonacci if $H_n = H_{n-1} + H_{n-2},\ n > 1$. Every Fibonacci sequence $\{H_n\}$ can be written as $H_n = A\alpha^n + B\beta^n$, where α,β are the roots of $x^2 - x - 1 = 0$. Thus

Theorem.

$$\sum_{i,j=0}^{n} a_{ij}H_iK_j = 0$$

for any two Fibonacci sequences if and only if

$$P(z,w) = \sum_{i,j=0}^{n} a_{ij}z^iw^j$$

vanishes on $\{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$.

Example. (Berzsenyi [1]): If n is even, prove that

$$\sum_{k=0}^{n} H_kK_{k+2m+1} = H_{m+n+1}K_{m+n+1} - H_{m+1}K_{m+1} + H_0K_{2m+1}\,.$$

The corresponding $P(z,w)$ is easily seen to satisfy the hypothesis of the theorem (using $\alpha\beta = -1$, $\alpha^2 - \alpha - 1 = 0$).

REFERENCE

1. G. Berzsenyi, "Sums of Products of Generalized Fibonacci Numbers," *The Fibonacci Quarterly,* Vol. 13 (1975), pp. 343–344.

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